AN ESSAY

ON THE

RESOLUTION OF EQUATIONS.

BY

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IN TWO PARTS.

PART I.

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CHAPTER I.

OF THE EXTENT TO WHICH THE METHOD OF TSCHIRNHAUSEN IS AVAILABLE FOR THE PURPOSE OF SOLVING EQUATIONS.

Preliminary Remarks and Propositions.

1. THE earliest uniform method for effecting the algebraical resolution of equations is, I believe, that proposed by Tschirnhausen in the Acts of Leipsic for the year 1683. It consists in transforming the general equation of the mth degree

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Ux + V = 0$$

into another equation of the same degree,

$$y^m + A'y^{m-1} + B'y^{m-2} + \ldots + U'y + V' = 0;$$

in which A', B', .. U' are severally to be made to vanish by assigning suitable values to P, Q, R, .. L in

$$y = \mathrm{P} + \mathrm{Q}x + \mathrm{R}x^2 + \ldots + \mathrm{L}x^{m-1},$$

which links any root of the equation in y to a corresponding root of the original equation in x. The point of view from which the problem of the resolution of equations is here contemplated is very different from that selected by later mathematicians. But the more unlike two modes of arriving at the same result are, the more instructive it will ever be to observe their approach and contact. I shall begin therefore with an examination of Tschirnhausen's method.

2. That the equation in y will be of the same degree as the one in x, we may easily convince ourselves from considering that it is capable of being resolved into precisely as many factors,

$$\begin{aligned}
&y-(\mathrm{P}+\mathrm{Q}x_1+\mathrm{R}x_1{}^2+\,..\,+\mathrm{L}x_1{}^{m-1}),\\
&y-(\mathrm{P}+\mathrm{Q}x_2+\mathrm{R}x_2{}^2+\,..\,+\mathrm{L}x_2{}^{m-1}),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&y-(\mathrm{P}+\mathrm{Q}x_m+\mathrm{R}x^2{}_m+\,..\,+\mathrm{L}x^{m-1}_m),
\end{aligned}$$

as there are roots, x_1, x_2, .. x_m, of the equation in x.

3. Again, from the well-known relations between the coëfficients and the roots

$$\begin{aligned}
A'&=-(y_1+y_2+y_3+\,..\,+y_m),\\
B'&=+(y_1y_2+y_1y_3+\,..\,y_{m-1}y_m),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
V'&=(-1)^m y_1y_2\,..\,y_m,
\end{aligned}$$

we may see that A', B', V' will, in the order of their occurrence, be of the first, the second, .. the nth degree relatively to P, Q, R, .. L; and lastly, that the roots of the proposed equation, x_1, x_2, .. x_m, will enter symmetrically into A', B', .. V'. All these results may, however, be embodied and made visible in an equation.

4. If, in fact, we suppose that

$$y_\tau=\mathrm{P}x_\tau^\alpha+\mathrm{Q}x_\tau^\beta+\mathrm{R}x_\tau^\gamma+\,..\,+\mathrm{L}x_\tau^\lambda,$$

where τ may have any one indifferently of the m values 1, 2, 3, .. m assigned to it, the equation in y will, as I proceed to show, be susceptible of the form

$$\begin{aligned}
&y^m-\\
&\mathfrak{S}\,.\,(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})y^{m-1}+\\
\frac{1}{1\,.\,2}&\mathfrak{S}\,.\,(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})^2y^{m-2}-\\
\frac{1}{1\,.\,2\,.\,3}&\mathfrak{S}\,.\,(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})^3y^{m-3}+\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
(-1)^m\frac{1}{1\,.\,2\,..\,m}&\mathfrak{S}\,.\,(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})^m=0\,;
\end{aligned}$$

α, β, γ, .. λ being any integers.

5. The functions here affected with $\mathfrak{S}$ are merely condensed modes of expression formed, in accordance with a known method of notation, by subjecting $\mathfrak{S}$, a symbol of operation, to the same laws, with certain obvious limitations, as would obtain if $\mathfrak{S}$ were a symbol of quantity. Thus by

$$\mathfrak{S}.(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})$$

is represented

$$\mathfrak{S}\alpha\ \mathrm{P}+\mathfrak{S}\beta\ \mathrm{Q}+\mathfrak{S}\gamma\ \mathrm{R}+\,..\,\mathfrak{S}\lambda\ \mathrm{L};$$

$\mathfrak{S}\alpha$, $\mathfrak{S}\beta$, .. $\mathfrak{S}\lambda$ being the sums of the αth, βth, .. λth powers respectively of the m quantities $x_1, x_2, .. x_m$. And, in general, if we expand

$$(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})^n$$

in precisely the same manner as we should do if we were operating on an ordinary algebraical expression, and then apply the symbol $\mathfrak{S}$ to each term, bearing in mind that $\alpha, \beta, \gamma, .. \lambda$ are the sole elements of the functions thus characterized by $\mathfrak{S}$, we shall arrive at the expression indicated by

$$\mathfrak{S}.(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+\,..\,+\lambda\mathrm{L})^n.$$

The expansion in question will accordingly consist of as many terms of the form

$$\frac{1.2..n\ \mathfrak{S}\alpha^a\beta^b..\lambda^l\ \mathrm{P}^a\mathrm{Q}^b..\mathrm{L}^l}{1.2..a\times1.2..b\times..\times1.2..l}$$

as there are different solutions in positive integers, or in zero of the equation

$$a+b+\,..\,+l=n.$$

$\mathfrak{S}\alpha^a\beta^b..\lambda^l$ is a symmetrical function of which

$$x_1^\alpha.x_2^\alpha...x_a^\alpha\times x_{1'}^\beta.x_{2'}^\beta...x_b^\beta\times..$$

is a term. It is what the general symmetrical function $\mathfrak{S}\alpha_1\alpha_2..\alpha_a\,\beta_1\beta_2..\beta_b..$, of which $x_1^{\alpha_1}.x_2^{\alpha_2}...x_a^{\alpha_a}\times x_{1'}^{\beta_1}.x_{2'}^{\beta_2}...x_b^{\beta_b}\times..$ is a term, will become when $\alpha_1=\alpha_2=..=\alpha$, $\beta_1=\beta_2=..=\beta$, ... Taking, for example, $\mathfrak{S}\alpha\beta$, and supposing it to be symmetrical with respect to two quantities t, u, we shall have

$$\mathfrak{S}\alpha\beta=t^\alpha u^\beta+u^\alpha t^\beta;$$

from which, on putting $\beta=\alpha$, there will be derived

$$\mathfrak{S}\alpha^2=t^\alpha u^\alpha+u^\alpha t^\alpha.$$

$\mathfrak{S}\alpha^2$ is thus composed of as many terms as $\mathfrak{S}\alpha\beta$. And a similar mode of derivation is supposed to extend to every order in the present system of symmetrical functions*.

6. It is not difficult to demonstrate the truth of this equation in y.

In effect, from the property of equations mentioned in 3, the coefficient of y^{m-n} in the transformed equation will be at once seen to be expressible by

$$(-1)^n \frac{1}{1.2..n} \Sigma y_1 y_2 y_3 .. y_n;$$

if we use the symbol Σ in an equally extended sense with $\mathfrak{S}$, that is to say, if we suppose $\Sigma y_1 y_2 y_3 .. y_n$ to be derivable from $\Sigma y_1^a y_2^b y_3^c .. y_n^l$ without any diminution in the number of its terms, on taking $a=b=c=..=l=1$.

But, according to the assigned form, the coefficient of y^{m-n} will be

$$(-1)^n \frac{1}{1.2..n} \mathfrak{S}.(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+..+\lambda\mathrm{L})^n;$$

n being any integer less than $m+1$.

Nothing therefore remains but to show that

$$\Sigma y_1 y_2 y_3 .. y_n = \mathfrak{S}.(\alpha\mathrm{P}+\beta\mathrm{Q}+\gamma\mathrm{R}+..+\lambda\mathrm{L})^n;$$

a theorem the truth of which may be proved as follows:—

Let us, in the first place, assume

$$\mathrm{R}=0, .. \mathrm{L}=0.$$

The expression for $\Sigma y_1 y_2 .. y_n$ will thus become

$$\Sigma(\mathrm{P}x_1^\alpha+\mathrm{Q}x_1^\beta)(\mathrm{P}x_2^\alpha+\mathrm{Q}x_2^\beta)..(\mathrm{P}x_n^\alpha+\mathrm{Q}x_n^\beta).$$

This function is, we perceive, of n dimensions relatively to P and Q; we see also that an αth power of each of the n quantities $x_1, x_2, .. x_n$ is successively joined to P, and a βth power to Q; hence we are led to conclude that

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* In the systems of symmetrical functions hitherto in use among mathematicians, $t^\alpha u^\alpha$, not $t^\alpha u^\alpha + u^\alpha t^\alpha$, would in the case we have been considering have the same characteristic as $t^\alpha u^\beta + u^\alpha t^\beta$. But had we thus *insulated* those symmetrical functions in which there are equal elements, we should not have arrived at a theorem in which the separation of the symbol $\mathfrak{S}$ from its subjects could have taken place.

$$\Sigma(Px_1^\alpha+Qx_1^\beta)(Px_2^\alpha+Qx_2^\beta)\ldots(Px_n^\alpha+Qx_n^\beta)$$
$$=v_0\,\mathfrak{S}\alpha^n\ P^n+v_1\,\mathfrak{S}\alpha^{n-1}\beta\ P^{n-1}Q$$
$$+v_2\,\mathfrak{S}\alpha^{n-2}\beta^2\ P^{n-2}Q^2+\ldots+v_n\,\mathfrak{S}\beta^n\ Q^n;$$

$v_0, v_1, v_2, \ldots v_n$ being certain constant but unknown quantities.

In order to determine these, let

$$\alpha=0, \quad \beta=0;$$

we shall then have

$$\Sigma(P+Q)^n x_1^0 x_2^0\ldots x_n^0=$$
$$\mathfrak{S}0^n(v_0P^n+v_1P^{n-1}Q+v_2P^{n-2}Q^2+\ldots+v_nQ^n).$$

And since, in general,

$$\Sigma(P+Q)^n x_1^0x_2^0\ldots x_n^0=(P+Q)^n\Sigma x_1^0x_2^0\ldots x_n^0=(P+Q)^n\mathfrak{S}0^n,$$

there will result, on making the requisite substitution,

$$(P+Q)^n=v_0P^n+v_1P^{n-1}Q+v_2P^{n-2}Q^2+\ldots+v_nQ^n.$$

Thus $v_0, v_1, v_2, \ldots v_n$ are respectively equal to $1, n, \frac{n(n-1)}{1.2}, \ldots 1$, the coefficients of the development of $(P+Q)^n$.

If, therefore, we introduce the equation of definition

$$\mathfrak{S}.(\alpha P+\beta Q)^n=$$
$$\mathfrak{S}\alpha^n\ P^n\ +n\,\mathfrak{S}\alpha^{n-1}\beta\ P^{n-1}Q\ +\frac{n(n-1)}{1.2}\,\mathfrak{S}\alpha^{n-2}\beta^2\ P^{n-2}Q^2+\ldots,$$

we shall finally obtain

$$\Sigma(Px_1^\alpha+Qx_1^\beta)(Px_2^\alpha+Qx_2^\beta)\ldots(Px_n^\alpha+Qx_n^\beta)=\mathfrak{S}.(\alpha P+\beta Q)^n.$$

And from this we can easily ascend to the general form in which R, .. L are not supposed to be equal to zero.

7. Wth respect to $\alpha, \beta, \gamma, \ldots \lambda$, I ought, however, to remark, that although the equation in y will subsist for all integral values of these quantities, yet if one or more of them exceed $m-1$, the functions P, Q, R, .. L will not be wholly indeterminate, but will be subject to certain conditions in consequence of the collapse of the series for y, as will appear hereafter.

Of the nature of the equations of condition which must be satisfied in bringing the equation in y *to the binomial form.*

8. We are now able to express $A', B', C', \ldots V'$, each of them in terms of P, Q, R, .. L, so as to exhibit the true character of

the equations of condition which must be satisfied in order that the equation in y may be reduced to

$$y^m + V' = 0,$$

or to

$$y^m + K'y^{m-n} + \ldots + V' = 0,$$

which becomes identical with the former when $n = m$.

For since, if, assigning to $\alpha, \beta, \gamma, \ldots \lambda$ the most simple values of which they are susceptible, we take

$$y = \mathrm{P} + \mathrm{Q}x + \mathrm{R}x^2 + \ldots + \mathrm{L}x^{n-1},$$

we shall have

$$K' = (-1)^n \frac{1}{1 \, . \, 2 \ldots n} \mathfrak{S} \, . \, \big(\mathbf{0}\mathrm{P} + \mathbf{1}\mathrm{Q} + \mathbf{2}\mathrm{R} + \ldots + (\mathbf{n-1})\mathrm{L}\big)^n;$$

it is manifest that the $(n-1)$ equations of condition for effecting the proposed reduction will be

$$\begin{aligned}
&\mathfrak{S} \, . \, \big(\mathbf{0}\mathrm{P} + \mathbf{1}\mathrm{Q} + \mathbf{2}\mathrm{R} + \ldots + (\mathbf{n-1})\mathrm{L}\big)^1 = 0,\\
&\mathfrak{S} \, . \, \big(\mathbf{0}\mathrm{P} + \mathbf{1}\mathrm{Q} + \mathbf{2}\mathrm{R} + \ldots + (\mathbf{n-1})\mathrm{L}\big)^2 = 0,\\
&\mathfrak{S} \, . \, \big(\mathbf{0}\mathrm{P} + \mathbf{1}\mathrm{Q} + \mathbf{2}\mathrm{R} + \ldots + (\mathbf{n-1})\mathrm{L}\big)^3 = 0,\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&\mathfrak{S} \, . \, \big(\mathbf{0}\mathrm{P} + \mathbf{1}\mathrm{Q} + \mathbf{2}\mathrm{R} + \ldots + (\mathbf{n-1})\mathrm{L}\big)^{n-1} = 0,
\end{aligned}$$

the numbers substituted for $\alpha, \beta, \gamma, \ldots \lambda$ being printed in a distinct type to remind us that they are to be associated exclusively with the symbol $\mathfrak{S}$.

9. Can we then solve these equations of condition? If we suppose $\mathrm{L} = 1$, and eliminate all but one of the quantities $\mathrm{P}, \mathrm{Q}, \mathrm{R}, \ldots$ we know by Bezout's theorem that we shall in general be conducted to a final equation of the $1 \, . \, 2 \ldots (n-1)$th degree. When, therefore, we have

$$1 \, . \, 2 \ldots (m-1) < m,$$

nothing can be more obvious than the way of applying Tschirnhausen's method. But how are we to proceed when

$$1 \, . \, 2 \ldots (m-1) > m,$$

which it always will be when m exceeds 3? We are thus led to distinguish two classes of transformations.

10. The most simple transformation of the class for which the product of the numbers marking the degrees of the equa-

tions of condition is less than m, depends on the solution of the problem

To EQUATE THE COEFFICIENT A' TO ZERO.

Here, in accordance with what has been proved in 8, we have $n-1=1$. Assuming, therefore,

$$y=\mathrm{P}+\mathrm{Q}x,$$

we may cause A' to vanish by assigning such values to P and Q as will satisfy the equations of the first degree

$$\mathrm{Q}=1,$$
$$\mathfrak{S}\,.\,(0\mathrm{P}+1\mathrm{Q})=0.$$

Thus when y can be found, x will be completely determined. In effect,

$$x=y-\mathrm{P},$$
$$\mathrm{P}=-\frac{\mathfrak{S}1}{\mathfrak{S}0}=(-1)^2\frac{A}{m},$$

as is evident.

Case of m$=2$. *Solution of quadratic equations. Reflections.*

11. In this way we arrive at a very comprehensive method of solving quadratic equations. For when $m=2$, the proposed equation will become

$$x^2+Ax+B=0,$$

and the equation in y will take the form

$$y^2+B'=0,$$

in which B' is a known function of A and B. Hence

$$x=-\mathrm{P}\pm\sqrt{-B'}.$$

12. The actual expressions for $-\mathrm{P}$ and $-B'$ might be obtained from the equations

$$-\mathrm{P}=(-1)^2\frac{\mathfrak{S}1}{\mathfrak{S}0}\,\mathrm{Q},$$

$$-B'=-\frac{1}{1\,.\,2}\mathfrak{S}\,.\,(0\mathrm{P}+1\mathrm{Q})^2,$$

on putting $\mathrm{Q}=1$. But if we avail ourselves of the knowledge which we already possess of the form of the expression for x, we may, by means of the principle of the equality of dimensions, elicit the results in question more rapidly as follows :—

Observing that the expressions for P and B' are rational and integral with respect to A and B, we see from the assigned form of the expression for x that we must have

$$x=\tau A\pm\sqrt{v_0A^2+v_1B};$$

τ, v_0, v_1 being certain numerical coefficients which will remain unaltered when we change the values of A and B.

It is very easy to determine these constants. For, taking $B=0$, the proposed equation will become

$$(x+A)x=0;$$

so that the expression

$$\tau A\pm\sqrt{v_0A^2}$$

must have the two values

$$-A,\quad 0.$$

Let then

$$\tau+\sqrt{v_0}=-1,$$

$$\tau-\sqrt{v_0}=0,$$

and we shall instantly perceive that

$$\tau=-\tfrac{1}{2},\quad v_0=\tfrac{1}{4}.$$

Lastly, to determine v_1, I take $A=0$. There must now be a coincidence between

$$\pm\sqrt{-B},$$

which is given immediately by the equation $x^2+B=0$, and

$$\pm\sqrt{v_1B},$$

to which the general expression is reduced when $A=0$. Hence

$$v_1=-1.$$

Substituting the numbers thus found for τ, v_0, v_1, in the expression for x, we recognize the well-known formula for the solution of quadratic equations.

13. The method which we have been considering points out very readily the more ancient method of solution by completing the square. For it gives at once

$$x+\mathrm{P}=\pm\sqrt{-B'};$$

so that on going back one step we come to

$$(x+\mathrm{P})^2=-B',$$

thus reaching the place from which we set out by the other method.

14. The next problem to be discussed is—

TO EQUATE THE TWO COEFFICIENTS A', B' SIMULTANEOUSLY TO ZERO.

The equations of condition here are

$$\mathfrak{S}\,.\,(0\mathrm{P}+1\mathrm{Q}+2\mathrm{R})^1=0,$$
$$\mathfrak{S}\,.\,(0\mathrm{P}+1\mathrm{Q}+2\mathrm{R})^2=0\,;$$

which, if $\mathrm{R}=1$, will evidently lead to a final equation in P or Q of the second degree. Could we therefore solve the equation in y, there would only remain, in order to find x, to determine the roots common to the equations

$$x^m+Ax^{m-1}+Bx^{m-2}+\,.\,.+V=0,$$
$$x^2+\mathrm{Q}x+\mathrm{P}-y=0,$$

when for y we substitute in succession each of its values,

$$y_1,\, y_2,\,.\,.\,y_m.$$

Development in the case of m = 3. *Solution of cubic equations. Comparison of the solution thus found with that of* Scipio Ferrei *and* Tartaglia. *Evolution of other methods.*

15. In the case of $m=3$, in which the proposed equation is

$$x^3+Ax^2+Bx+C=0,$$

we see that the transformed equation in y will assume the solvable form

$$y^3+C'=0.$$

If therefore we combine

$$x^3+Ax^2+Bx+C=0,$$
$$x^2+\mathrm{Q}x+\mathrm{P}-y=0,$$

we shall be conducted, by the method of the highest common divisor or otherwise, to an equation of the first degree in x,

$$\mathrm{M}x+\mathrm{N}=0\,;$$

in which M and N are in general determinate non-evanescent functions of the known quantities P, Q, y. In effect*,

$$\mathrm{M}=B-(\mathrm{P}-y)-(A-\mathrm{Q})\mathrm{Q},$$
$$\mathrm{N}=C-(A-\mathrm{Q})(\mathrm{P}-y).$$

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* Denoting by $\mathrm{X}_3=0$, $\mathrm{X}_2=0$ the two equations in x, the degrees of which are marked by the numbers 3, 2 respectively, we may manifestly assume

$$\mathrm{X}_3=\mathrm{K}\mathrm{X}_2+\mathrm{R},$$

where

$$\mathrm{K}=x+\alpha,\quad \mathrm{R}=\mathrm{M}x+\mathrm{N}.$$

For, on replacing X_2 by $x^2+\mathrm{Q}x+\mathrm{P}-y$, there will arise the following

Accordingly, on designating by M_1, N_1, by M_2, N_2, and by M_3, N_3 the abbreviated expressions which take the places of M, N when we write y_1, y_2, y_3 successively for y, the three roots of the proposed cubic equation will be represented by

$$-\frac{N_1}{M_1},\quad -\frac{N_2}{M_2},\quad -\frac{N_3}{M_3}.$$

It will be instructive to compare these expressions for the roots with the older ones discovered by Scipio Ferrei and Tartaglia, which, it will be remembered, are integral with respect to the coefficients of the proposed equation in x.

16. With this view, I shall first show that $-\frac{N_1}{M_1}, -\frac{N_2}{M_2}, -\frac{N_3}{M_3}$, which, in the order of their occurrence, are rational functions of y_1, y_2, y_3, admit of integral forms in relation to these quantities.

To express symbolically that M_n and N_n are functions of y_n, n being susceptible of any one indifferently of the three values 1, 2, 3, let

$$M_n=\phi(y_n),\quad N_n=\chi(y_n);$$

we shall then have, by following a known method, the very source indeed of which we seem here to have reached*,

$$\frac{N_1}{M_1}=\frac{\chi(y_1)}{\phi(y_1)}=\frac{\chi(y_1)\ \phi(y_2)\ \phi(y_3)}{\phi(y_1)\ \phi(y_2)\ \phi(y_3)}.$$

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expression for X_3 :—

$$\begin{array}{r|r|r|} x^3+Q & x^2+P-y & x \\ +\alpha & +\alpha Q & +\alpha(P-y) \\ & +M & +N \end{array}$$

which, since there are three indeterminate quantities α, M, N, can be made to coincide throughout its extent with

$$x^3+Ax^2+Bx+C,$$

the first member of the proposed equation.

In what precedes, it is implied that when $R=0$, $X_3=0$, we shall also have $X_2=0$. But of the circumstances under which $K=0$, and of the relations which must exist among the coefficients A, B, C, in order that the expressions for the roots $-\frac{N_1}{M_1}$, $-\frac{N_2}{M_2}$, $-\frac{N_3}{M_3}$ may, each of them, take the form $\frac{0}{0}$, I reserve the consideration for another place.

* See the reflections of Lagrange on the algebraical resolution of equations, in the Memoirs of the Berlin Academy of Sciences for the years 1770 and 1771.

Again, the product $\phi(y_2)\,\phi(y_3)$ is a whole and symmetrical function of y_2, y_3. Putting, therefore,

$$(y-y_2)(y-y_3)=y^2+ty+u,$$

we see that $\phi(y_2)\,\phi(y_3)$ will be expressible by a whole function of t, u; and consequently, as

$$(y-y_1)(y^2+ty+u)=(y-y_1)(y-y_2)(y-y_3)=y^3+C',$$

by a whole function of y_1.

We shall accordingly have

$$\frac{N_1}{M_1}=\frac{\chi(y_1)\;\psi(y_1)}{\mathbf{H}};$$

ψ as well as χ being characteristic of an integral function of y_1, and **H** being symmetrical with respect to all the roots of the equation in y, that is to say, rational with respect to C'.

In like manner we should find

$$\frac{N_2}{M_2}=\frac{\chi(y_2)\;\psi(y_2)}{\mathbf{H}},$$

$$\frac{N_3}{M_3}=\frac{\chi(y_3)\;\psi(y_3)}{\mathbf{H}}.$$

Whence it appears that

$$-\frac{N_1}{M_1},\quad -\frac{N_2}{M_2},\quad -\frac{N_3}{M}$$

may be brought to the forms

$$\omega(y_1),\quad \omega(y_2),\quad \omega(y_3);$$

in which ω is characteristic of a function rational and integral with respect to y_n.

17. The expressions at which we have just arrived lead directly to the forms treated of by Euler and Bezout; but without stopping to consider these, I proceed to trace the further changes of form which $\omega(y_1)$, $\omega(y_2)$, $\omega(y_3)$ must undergo in order to coincide, on taking $A=0$, with the roots of the equation $x^3+Bx+C=0$, as given by the Italian geometers:

$$\sqrt[3]{K+\sqrt{\Lambda}}+\sqrt[3]{K-\sqrt{\Lambda}},$$

$$\iota\sqrt[3]{K+\sqrt{\Lambda}}+\iota^2\sqrt{K-\sqrt{\Lambda}},$$

$$\iota^2\sqrt[3]{K+\sqrt{\Lambda}}+\iota\sqrt[3]{K-\sqrt{\Lambda}};$$

K, Λ being certain rational and integral functions of the coefficients B, C; and 1, ι, ι^2 the roots of the binomial equation $\rho^3-1=0$.

18. Now $\omega(y_n)$ is composed of terms of the form gy_n^ν, where ν is a positive integer, and g, for a fixed value of ν, is a determinate function of B, C, not of course subject to the conditions of being whole and rational. Again, since $y_n^4=-C'y_n$, $y_n^5=-C'y_n^2$, . ., it is manifest that for the series of terms $g_1\,y_n^{\nu_1}$, $g_2\,y_n^{\nu_2}$, . . which first present themselves, we may substitute another series in which none of the exponents of y_n shall exceed 2. We are thus conducted to the equation

$$\omega(y_n)=\mu_2+\mu_1\,y_n+\mu_0\,y_n^2\,;$$

where μ_2, μ_1, μ_0 are all of them known algebraical functions of B, C, not involving y_n.

From this, on putting, as is permitted,

$$\omega(y_n)=x_n,\quad y_n=\iota^{n-1}\sqrt[3]{-C'},$$

there will result

$$x_1=\mu_2+\mu_1\sqrt[3]{-C'}+\mu_0\sqrt[3]{(-C')^2},$$
$$x_2=\mu_2+\iota\mu_1\sqrt[3]{-C'}+\iota^2\mu_0\sqrt[3]{(-C')^2},$$
$$x_3=\mu_2+\iota^2\mu_1\sqrt[3]{-C'}+\iota\mu_0\sqrt[3]{(-C')^2}$$

and thence, since $0=-A=x_1+x_2+x_3=3\mu_2$, the expressions

$$\sqrt[3]{\mu_1{}^3(-C')}+\sqrt[3]{\mu_0{}^3(-C')^2},$$
$$\iota\sqrt[3]{\mu_1{}^3(-C')}+\iota^2\sqrt[3]{\mu_0{}^3(-C')^2},$$
$$\iota^2\sqrt[3]{\mu_1{}^3(-C')}+\iota\sqrt[3]{\mu_0{}^3(-C')^2}\,;$$

which already bear, as we see, very marked resemblances in form to those in 17.

19. The question, however, is not, Are the two sets of roots of $x^3+Bx+C=0$ capable of becoming ultimately identical, but rather, By the evolution of what properties does such identity manifest itself. Let us then examine what takes place at the points of contact.

20. Reverting to the three equations in 18 for x_1, x_2, x_3, we deduce

$$x_1+x_2+x_3=3\mu_2,$$
$$x_1+\iota^2x_2+\iota x_3=3\mu_1\sqrt[3]{-C'},$$
$$x_1+\iota x_2+\iota^2x_3=3\mu_0\sqrt[3]{(-C')^2}.$$

Now these give, besides $\mu_2=0$,

$$\mu_1{}^3(-C')=\left(\frac{1}{3}\right)^3(x_1+\iota^2x_2+\iota x_3)^3,$$

$$\mu_0{}^3(-C')^2=\left(\frac{1}{3}\right)^3(x_1+\iota x_2+\iota^2x_3)^3.$$

Hence, in order that the requisite identity may subsist, the two functions

$$\left(\frac{1}{3}\right)^3(x_1+\iota^2x_2+\iota x_3)^3,\quad \left(\frac{1}{3}\right)^3(x_1+\iota x_2+\iota^2x_3)^3,$$

which are whole and rational with respect to x_1, x_2, x_3, must, when $\omega(y_n)$ and y_n take the prescribed expressions, be respectively equal to

$$\mathrm{K}+\sqrt{\Lambda},\quad \mathrm{K}-\sqrt{\Lambda};$$

and must consequently admit of becoming the roots of a quadratic equation,

$$\{z-(\mathrm{K}+\sqrt{\Lambda})\}\{z-(\mathrm{K}-\sqrt{\Lambda})\}=0;$$

the coefficients of which will, as well as K and Λ, be rational and integral functions of B and C.

21. With respect to K and Λ, we might, as before, by applying the principle of the equality of dimensions, find that in the present case

$$\mathrm{K}=-\tfrac{1}{2}C,\quad \Lambda=\tfrac{1}{4}C^2+\tfrac{1}{27}B^3,$$

precisely as they ought to be.

And by the guidance of the same principle we might also obtain the expressions for K and Λ when $A>$ or <0, and when consequently μ_2 would have to be retained in each root.

But having in 12 fully explained the nature of the method to be pursued, I shall pass on to the problem involving the resolution of equations of the fourth degree.

22. We have now reached the point at which

$$1.2..(m-1)>m;$$

m being here equal to 4.

The solution of biquadratic equations may, however, be made to depend on that of the problem—

To REDUCE THE EQUATION

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + Dx^{m-4} + \ldots + V = 0$$

TO THE FORM

$$y^m - B'y^{m-2} + D'y^{m-4} + \ldots + V' = 0;$$

as is manifest.

The equations of condition for effecting the proposed reduction are

$$\mathfrak{S} . (0\mathrm{P} + 1\mathrm{Q} + 2\mathrm{R})^1 = 0,$$
$$\mathfrak{S} . (0\mathrm{P} + 1\mathrm{Q} + 2\mathrm{R})^3 = 0;$$

for which

$$1 . 3 < m.$$

We thus see that if we designate by

$$x_n = \mu_{m-1} + \mu_{m-2} y_n + \mu_{m-3} y_n^2 + \ldots + \mu_0 y_n^{m-1},$$

the equation to which we shall be conducted, as in the preceding investigation, by means of

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots + V = 0,$$
$$x^2 + \mathrm{Q}x + \mathrm{P} - y = 0,$$

the coefficients $\mu_{m-1}, \mu_{m-2}, \ldots \mu_0$ will, if $\mathrm{R} = 1$, ultimately depend on an equation of the third degree in P or Q. To fix our ideas, say Q. All the coefficients in question may therefore in general be considered as known quantities. If therefore $y_1, y_2, \ldots y_m$ be also known, as they will be when $m = 4$, the expressions for x_1, $x_2, \ldots x_m$ will be completely determined.

Case of m = 4. *Approach to* Euler's *method. Ultimate forms assumed by the expressions for the roots. Evolution of the method of* Descartes *and that of* Louis Ferrari. *Method of* Lagrange *and* Vandermonde.

23. When $m = 4$, the equation in y will, as has been already intimated, take the form

$$y^4 + B'y^2 + D' = 0,$$

and will consequently admit of being solved as a quadratic equation. Nothing therefore remains in order to obtain the roots of the biquadratic equation

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0,$$

but to put $m=4$ in the general expression for x_n, assigning successively to n the four values 1, 2, 3, 4. Accordingly, we shall find

$$\begin{aligned}
x_1&=\mu_3+\mu_2 y_1+\mu_1 y_1^2+\mu_0 y_1^3,\\
x_2&=\mu_3+\mu_2 y_2+\mu_1 y_2^2+\mu_0 y_2^3,\\
x_3&=\mu_3+\mu_2 y_3+\mu_1 y_3^2+\mu_0 y_3^3,\\
x_4&=\mu_3+\mu_2 y_4+\mu_1 y_4^2+\mu_0 y_4^3.
\end{aligned}$$

24. The solution of Euler has here come into view. I shall, however, be forced to postpone what I have to say with respect to that solution until, at all events, I have discussed the ultimate forms which x_1, x_2, x_3, x_4 assume in coinciding with the more ancient expressions for the roots of an equation of the fourth degree.

25. On reflecting that the four values of y may be represented by

$$\pm\sqrt{\Theta+\sqrt{\mathrm{I}}},\quad \pm\sqrt{\Theta-\sqrt{\mathrm{I}}},$$

we shall instantly perceive that

$$\mu_3+\mu_2 y+\mu_1 y^2+\mu_0 y^3,$$

or, which is the same thing,

$$\begin{aligned}&\mu_3+\mu_1 y^2\\ &+(\mu_2+\mu_0 y^2)y\end{aligned}$$

will, if we denote

$$\begin{array}{llll}\mu_3+\mu_1\Theta, & \mu_1, & \mu_2+\mu_0\Theta, & \mu_0,\end{array}$$

by

$$\begin{array}{llll}a, & b, & c, & d,\end{array}$$

respectively, lead to the following forms for the roots of a biquadratic equation:—

$$\left.\begin{aligned}
&a+b\sqrt{\mathrm{I}}+(c+d\sqrt{\mathrm{I}})\sqrt{\Theta+\sqrt{\mathrm{I}}},\\
&a+b\sqrt{\mathrm{I}}-(c+d\sqrt{\mathrm{I}})\sqrt{\Theta+\sqrt{\mathrm{I}}},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&a-b\sqrt{\mathrm{I}}+(c-d\sqrt{\mathrm{I}})\sqrt{\Theta-\sqrt{\mathrm{I}}},\\
&a-b\sqrt{\mathrm{I}}-(c-d\sqrt{\mathrm{I}})\sqrt{\Theta-\sqrt{\mathrm{I}}},
\end{aligned}\right\}$$

a, b, c, d, Θ, I, all of them, in general, admitting of being expressed as determinate rational functions of Q.

26. Such are the ultimate forms to which we are in the pre-

sent instance conducted by Tschirnhausen's method. That the coincidence spoken of in 24 can now take place, may, I think, be shown in the most striking manner by actually evolving from them, as if still undiscovered, the methods previously devised by mathematicians for the solution of the same problem.

27. Let x_1 and x_2 denote the first pair of the roots in question, we shall accordingly have

$$\frac{x_1-(a+b\sqrt{\text{I}})}{c+d\sqrt{\text{I}}}=+\sqrt{\Theta+\sqrt{\text{I}}}$$

$$\frac{x_2-(a+b\sqrt{\text{I}})}{c+d\sqrt{\text{I}}}=-\sqrt{\Theta+\sqrt{\text{I}}};$$

wherein x_1 and x_2 would be similarly involved if the signs of the second members were alike.

If, therefore, we square each member of either of these equations, and designate by

$$x^2+\text{L}'x+\text{M}'=0$$

the equation which will thence arise, this will manifestly have for its roots the two quantities x_1 and x_2.

In like manner, if we consider the second pair x_3 and x_4, we shall find that they admit of being included as roots in the same quadratic equation

$$x^2+\text{L}''x+\text{M}''=0,$$

L'' and M'' differing from L' and M' merely in the sign of the radical $\sqrt{\text{I}}$.

It follows, therefore, from what has been said with respect to a, b, c, d, Θ, I, that the biquadratic equation

$$x^4+Ax^3+Bx^2+Cx+D=0$$

is decomposable into two quadratic equations without the aid of any equation of a degree higher than the third.

This is the method of solution discovered by the illustrious Descartes.

28. Further, if we observe that L', M', L'', M'' are such that we may take

$$\text{L}'=p+q\sqrt{\text{I}},\quad \text{M}'=r+s\sqrt{\text{I}},$$

$$\text{L}''=p-q\sqrt{\text{I}},\quad \text{M}''=r-s\sqrt{\text{I}},$$

p, q, r, s being, in general, determinate rational functions of Q, we shall have no difficulty in ascending to the most ancient method, that of Louis Ferrari, which consists in bringing the equation of the fourth degree to the form

$$(x^2+\alpha x+\beta)^2-\gamma(x+\delta)^2=0;$$

where α, β, γ, δ do not involve x.

Indeed, this equation resolving itself into

$$x^2+\alpha x+\beta=+\sqrt{\gamma}(x+\delta),$$

$$x^2+\alpha x+\beta=-\sqrt{\gamma}(x+\delta);$$

we may see at a glance what values must be assigned to α, β, γ, δ, in order that the two systems may coincide.

29. With no less facility may we evolve directly from the expressions in 25 the properties of those non-symmetrical functions of the roots

$$x_1x_2+x_3x_4, \quad (x_1+x_2-x_3-x_4)^2,$$

which presented themselves to Lagrange and Vandermonde while separately engaged in deriving from the theory of combinations, a method that appeared subsequently to the time of Tschirnhausen.

30. We perceive that each pair of roots in 25 is composed of functions of the forms

$$t+u, \quad t-u,$$

and that therefore either of the products

$$x_1x_2, \quad x_3x_4,$$

will be of the form

$$t^2-u^2.$$

In effect,

$$x_1x_2=(a+b\sqrt{\text{I}})^2-(\Theta+\sqrt{\text{I}})(c+d\sqrt{\text{I}})^2,$$

$$x_3x_4=(a-b\sqrt{\text{I}})^2-(\Theta-\sqrt{\text{I}})(c-d\sqrt{\text{I}})^2;$$

or, as is manifest,

$$x_1x_2=v+w\sqrt{\text{I}},$$

$$x_3x_4=v-w\sqrt{\text{I}},$$

v, w being rational and integral functions of a, b, c, d, Θ, I; whence results

$$x_1x_2+x_3x_4=2v.$$

Thus we see that the non-symmetrical function $x_1x_2+x_3x_4$

may be expressed as a rational function of Q, and may consequently be determined by solving an equation of the third degree.

31. Again, as

$$x_1+x_2=2a+2b\sqrt{\mathrm{I}},$$
$$x_3+x_4=2a-2b\sqrt{\mathrm{I}},$$

we find by subtraction,

$$x_1+x_2-x_3-x_4=4b\sqrt{\mathrm{I}};$$

from which, if we square both sides to free the equation from the radical sign, there will at once spring

$$(x_1+x_2-x_3-x_4)^2=4^2b^2\mathrm{I}.$$

We have therefore arrived at another non-symmetrical function of the roots which is expressible by a rational function of Q.

32. It is clear that the number of such functions might be increased. But I must now proceed to the second class of transformations mentioned in 9.

33. Tschirnhausen, and after him Euler, took, as is well known, a series in y which did not involve any power of x higher than the $(m-1)$th; m denoting the degree of the equation in x. They did this to avoid the collapse of the series which would take place were a power of x in it to equal or exceed the highest power of x in the original equation. No opening was therefore immediately visible to them through which to discern a general mode of counteracting the effect of the elevation in degree of the final equation when

$$1.2\,..\,(m-1)>m.$$

Is it certain, however, that an insurmountable barrier is opposed to the introduction of more than the prescribed number of available indeterminates into the series for y? Is there no way through the collapse?

34. Let us consider the problem:

TO TAKE AWAY THE SECOND, THIRD, AND FOURTH TERMS AT ONCE FROM THE GENERAL EQUATION OF THE mTH DEGREE.

Supposing that

$$y=\mathrm{P}+\mathrm{Q}x+\mathrm{R}x^2+\,..\,+\mathrm{L}x^{\lambda},$$

we shall have for the equations of condition,

$$\begin{aligned}
&\mathfrak{S}\,.\,(0P+1Q+2R+\,..\,+\lambda L)^1=0,\\
&\mathfrak{S}\,.\,(0P+1Q+2R+\,..\,+\lambda L)^2=0,\\
&\mathfrak{S}\,.\,(0P+1Q+2R+\,..\,+\lambda L)^3=0,
\end{aligned}$$

which lead, as we see, to a final equation of 1 . 2 . 3 or 6 dimensions. Here then we seem to be stopped. But when the series for y rises to the fourth power of x, the difficulty may be eluded in the following manner.

Mode of solution when $\lambda=4$.

35. Since in the equation for x we are permitted to assume $A=0$, $B=0$, it is clear that $\mathfrak{S}1$ and $\mathfrak{S}1^2$ may both of them be made to vanish*. Hence on considering that the first two of the equations of condition may, when $\lambda=4$, take the forms

$$\begin{gathered}
\mathfrak{S}1\;Q+\\
\mathfrak{S}\,.\,(0P+2R+3S+4T)^1=0,
\end{gathered}$$

$$\begin{gathered}
\mathfrak{S}1^2\;Q^2+\\
2\mathfrak{S}1\,.\,(0P+2R+3S+4T)^1\;Q+\\
\mathfrak{S}\,.\,(0P+2R+3S+4T)^2=0,
\end{gathered}$$

we shall perceive that Q may be completely detached from both these equations if we determine P, R, S, T so that

$$\begin{aligned}
&\mathfrak{S}\,.\,(0P+2R+3S+4T)^1=0,\\
&\mathfrak{S}1\,.\,(0P+2R+3S+4T)^1=0,\\
&\mathfrak{S}\,.\,(0P+2R+3S+4T)^2=0,
\end{aligned}$$

where the product of the numbers which mark the dimensions relatively to P, R, S, T is only 1 . 1 . 2 or 2.

In this way, without resolving any equation of a higher degree than the second, we shall have

$$\begin{aligned}
A'&=0Q+0,\\
B'&=0Q^2+0Q+0.
\end{aligned}$$

And Q, which as yet, therefore, is wholly undetermined, may now satisfy the cubic equation

$$\mathfrak{S}\,.\,(0P+1Q+2R+3S+4T)^3=0;$$

---

* See Articles 3, 5.

that is,

$$\begin{gathered}\mathfrak{S}1^3\,Q^3+\\ 3\mathfrak{S}1^2\,.\,(0P+2R+3S+4T)^1\,Q^2+\\ 3\mathfrak{S}1\,.\,(0P+2R+3S+4T)^2\,Q+\\ \mathfrak{S}\,.\,(0P+2R+3S+4T)^3=0\,;\end{gathered}$$

thus fulfilling the third and last condition,

$$C'=0.$$

36. It may easily be shown that P, Q, R, S, T, y will none of them in general assume the form

$$\frac{\mathbf{G}}{(1-1)\mathbf{H}},$$

where $\mathbf{G}$ and $\mathbf{H}$ are integral functions of the $(m-2)$ arbitrary coefficients $C, D, \ldots V$.

Returning to the equations in P, R, S, T, we see that the first of them will be reducible to

$$\mathfrak{S}\,.\,(0P+3S+4T)=0,$$

since $\mathfrak{S}2=0$. We may also perceive that the second equation of the group will become

$$\mathfrak{S}\,.\,(3R+4S+5T)=0.$$

For since $\mathfrak{S}\tau\upsilon=\mathfrak{S}\tau\mathfrak{S}\upsilon-\mathfrak{S}(\tau+\upsilon)$[*], it is evident that the coefficients of P, R, S, T in

$$\mathfrak{S}1\,.\,(0P+2R+3S+4T)$$

may all of them be derived from the expression

$$\mathfrak{S}1\mathfrak{S}\upsilon-\mathfrak{S}(1+\upsilon),$$

on taking υ successively equal to $0, 2, 3, 4$; whence

$$\mathfrak{S}1\,.\,(0P+2R+3S+4T)=-\mathfrak{S}\,.\,(3R+4S+5T),$$

$\mathfrak{S}1$ being equal to zero.

---

[*] The truth of this proposition will be manifest if we bear in mind that $\mathfrak{S}\tau\upsilon$, $\mathfrak{S}\tau$, $\mathfrak{S}\upsilon$, $\mathfrak{S}(\tau+\upsilon)$ are the symmetrical functions composed of terms of the forms $t^\tau u^\upsilon$, t^τ, t^υ, $t^{\tau+\upsilon}$, respectively. In effect,

$$\begin{gathered}(t^\tau+u^\tau+\ldots)(t^\upsilon+u^\upsilon+\ldots)\\ =\\ t^{\tau+\upsilon}+u^{\tau+\upsilon}+\ldots\\ +t^\tau u^\upsilon+u^\tau t^\upsilon+\ldots\end{gathered}$$

Accordingly we shall have

$$P = -\frac{1}{\mathfrak{S}0}\mathfrak{S}.(3S+4T),$$

$$R = -\frac{1}{\mathfrak{S}3}\mathfrak{S}.(4S+5T).$$

We conclude, therefore, since we may assume $T=1$, that neither P nor R will be of the form $\frac{G}{(1-1)H}$, unless S be of that form*.

Substituting now these expressions for P and R in the equation

$$\mathfrak{S}.(0P+2R+3S+4T)^2=0,$$

the first member of which is an integral function of P, R, S, T, we shall obviously be conducted to a quadratic equation in S with determinate coefficients. This equation I shall represent by

$$\alpha S^2+2\beta ST+\gamma T^2=0;$$

α, β, γ being certain rational functions of $C, D, \ldots V$, free from evanescent denominators. The expression for S will consequently be

$$S = -\frac{\beta \pm \sqrt{\beta^2-\alpha\gamma}}{\alpha}T.$$

Unless, then, α be equal to zero, S will not take the form $\frac{G}{(1-1)H}$.

Now since, in order to obtain α, we need not consider the whole development of the function $\mathfrak{S}.(0P+2R+3S+4T)^2$, but only that part of it which is affected with S^2, it is clear that if we assume

$$P=pS+p'T,$$
$$R=rS+r'T,$$

assigning to p, p', r, r' such values as are deducible from the expressions previously found for P and R, we shall obtain α by merely writing p and r for P and R respectively, 1 for S, and suppressing the term 4T. Hence

$$\alpha=\mathfrak{S}.(0p+2r+3s)^2,$$

where

$$p=-\frac{\mathfrak{S}3}{\mathfrak{S}0},\quad r=-\frac{\mathfrak{S}4}{\mathfrak{S}0},\quad s=1.$$

---

* The exceptional case of $C=0$ will be considered hereafter.

Further, if we expand $\mathfrak{S}.(0p+2r+3s)^2$ according to the descending powers of s, the expression for α will become

$$\mathfrak{S}3^2 s^2 + 2\mathfrak{S}3.(0p+2r)s + \mathfrak{S}.(0p+2r)^2;$$

or since $\mathfrak{S}0=m$, $\mathfrak{S}2=0$,

$$\begin{gathered}\{(\mathfrak{S}3)^2-\mathfrak{S}6\}s^2+\\ 2\{(m-1)\mathfrak{S}3\,p+(-\mathfrak{S}5)r\}s+\\ m(m-1)p^2+(-\mathfrak{S}4)r^2;\end{gathered}$$

from which, on eliminating p, r, s, there will finally result

$$\alpha=\frac{1}{m}(\mathfrak{S}3)^2+2\frac{\mathfrak{S}4\mathfrak{S}5}{\mathfrak{S}3}-\frac{(\mathfrak{S}4)^3}{(\mathfrak{S}3)^2}-\mathfrak{S}6;$$

α cannot therefore vanish without inducing a relation among the coefficients $C, D, \ldots V$.

Having thus shown that S will not assume the form $\frac{\mathbf{G}}{(1-1)\mathbf{H}}$, it will immediately be seen that Q and y, as well as P and R, will, exclusively of particular cases, be all of them determinate in value. But before proceeding to the collapse, I wish to say a few words on equations of the fifth degree.

Remarkable forms to which the general equation of the fifth degree is reducible.

37. It is evident that we may take for the transformed equation not only

$$y^m+0+0+0+D'y^{m-4}\ldots+U'y+V=0,$$

but also

$$y^m+0+0+C'y^{m-3}+0+E'y^{m-5}\ldots+U'y+V'=0;$$

since the method we have been considering, which consists in detaching Q from A' and B', will enable us to satisfy $A'=0$, $B'=0$, $D'=0$, by merely substituting the biquadratic equation $D'=0$ for the cubic $C'=0$, as our final equation for determining Q.

Again, from these two equations in y there will spring, on putting $y=\frac{1}{z}$, two other forms,

$$z^m+\frac{U'}{V'}z^{m-1}\ldots+\frac{D'}{V'}z^4+0+0+0+\frac{1}{V'}=0,$$

$$z^m+\frac{U'}{V'}z^{m-1}\ldots+0+\frac{C'}{V'}z^3+0+0+\frac{1}{V'}=0.$$

Whence it follows that the solution of the general equation of the fifth degree reduces itself to that of any one of the trinomial equations*,

$$
\begin{aligned}
&x^5 + Dx + E = 0,\\
&x^5 + Cx^2 + E = 0,\\
&x^5 + Bx^3 + E = 0,\\
&x^5 + Ax^4 + E = 0\,;
\end{aligned}
$$

a result of such a character as may well shake our trust in the validity of the conclusion come to by mathematicians of the present day,—that it is in general not possible to solve algebraically equations of the fifth degree.

Collapse of the series for y *in the case of* $\mathrm{m}=4$.

38. Let us now observe what takes place whem $m=4$.

The series for y may here be reduced to

$$y = \mathrm{P}' + \mathrm{Q}'x + \mathrm{R}'x^2 + \mathrm{S}'x^3\,;$$

in which

$$
\begin{aligned}
&\mathrm{P}' = \mathrm{P} - \mathrm{T}D, \quad && \mathrm{Q}' = \mathrm{Q} - \mathrm{T}C,\\
&\mathrm{R}' = \mathrm{R}, && \mathrm{S}' = \mathrm{S}\,;
\end{aligned}
$$

since the proposed equation in x gives $x^4 = -Cx - D$.

The equations of condition $A'=0$, $B'=0$, $C'=0$, will accordingly become, if $-\mathrm{T}C = q$,

$$
\begin{aligned}
&\mathfrak{S}\,.\,(0\mathrm{P}' + 1[\mathrm{Q}+q] + 2\mathrm{R}' + 3\mathrm{S}')^1 = 0,\\
&\mathfrak{S}\,.\,(0\mathrm{P}' + 1[\mathrm{Q}+q] + 2\mathrm{R}' + 3\mathrm{S}')^2 = 0,\\
&\mathfrak{S}\,.\,(0\mathrm{P}' + 1\mathrm{Q}' + 2\mathrm{R}' + 3\mathrm{S}')^3 = 0.
\end{aligned}
$$

Hence as Q cannot be detached from the first and second of these equations, by the method in 35, without being accompanied by q, which is similarly involved, we shall find, in arriving at the forms

$$
\begin{aligned}
&0\mathrm{Q} + 0 = 0,\\
&0\mathrm{Q}^2 + 0\mathrm{Q} + 0 = 0,
\end{aligned}
$$

that we shall be conducted to three homogeneous equations between the three quantities P′, R′, S′.

---

* The first of these trinomial equations appeared in Part II. of my 'Mathematical Researches,' in the year 1834; the second, third, and fourth in a subsequent Part. They are now well known. See Sir W. R. Hamilton's Inquiry on the subject in the Sixth Report of the British Association for the Advancement of Science: also M. Serret's 'Cours d'Algèbre Supérieure,' Note V.

We must therefore have in general

$$P'=0, \quad R'=0, \quad S'=0.$$

The equation $C'=0$ will now give, except when $\mathfrak{S}1^3=0$,

$$Q'=0.$$

Lastly, from $y^4+D'=0$ there will result

$$y=0.$$

Thus the equation between x and y,

$$y=P'+Q'x+R'x^2+S'x^3,$$

will, unless the coefficients C, D be subject to certain conditions, resolve itself into

$$0=0+0x+0x^2+0x^3.$$

39. We may verify this, by showing directly from the expressions obtained in 36 for P, R, S, that when $m=4$ the series

$$P+Qx+Rx^2+Sx^3+Tx^4$$

will become a multiple of the evanescent function

$$D+Cx+x^4.$$

In effect, α, 2β, γ, the coefficients of the quadratic equation between S and T, may, as is manifest, be universally defined by

$$\alpha=\mathfrak{S}\,.\,(0p+2r+3s+4t)^2,$$
$$\beta=\mathfrak{S}\,.\,(0p+2r+3s+4t)(0p'+2r'+3s+4t'),$$
$$\gamma=\mathfrak{S}\,.\,(0p'+2r'+3s'+4t')^2.$$

With respect to the quantities here involved, p, r, s are already known; p', r' differ from p, r, by having in their numerators $\mathfrak{S}4$, $\mathfrak{S}5$, for $\mathfrak{S}3$, $\mathfrak{S}4$, respectively; t' like s is equal to 1; and t, s', which have been introduced merely for the purpose of exhibiting the symmetry of the calculus*, are both of them equal to zero.

---

* We may see beforehand, without going through any calculations, that in the coefficient of S^2 the quantities p, r, s, t must be involved in precisely the same manner as are P, R, S, T in the original function. A similar remark with respect to p', r', s', t', is applicable to the coefficient of T^2; while the coefficient of ST must be such as, abstractedly of the numerical coefficient 2, to coincide indifferently with the coefficient of S^2 or that of T^2, when p', r', s', t' are equal to p, r, s, t in the order of their occurrence. Indeed we may easily see, that if in any rational and integral function of

From what has been proved of α, it is clear indeed that we must have

$$\begin{gathered}\alpha=\\ m^{-1}\{(\mathfrak{S}3)^2-m\mathfrak{S}6\}+\\ 2(\mathfrak{S}3)^{-1}\mathfrak{S}4\mathfrak{S}5-\\ (\mathfrak{S}3)^{-2}(\mathfrak{S}4)^3,\end{gathered}$$

$$\begin{gathered}\beta=\\ m^{-1}\{\mathfrak{S}3\mathfrak{S}4-m\mathfrak{S}7\}+\\ (\mathfrak{S}3)^{-1}\{\mathfrak{S}4\mathfrak{S}6+(\mathfrak{S}5)^2\}-\\ (\mathfrak{S}3)^{-2}(\mathfrak{S}4)^2\mathfrak{S}5,\end{gathered}$$

$$\begin{gathered}\gamma=\\ m^{-1}\{(\mathfrak{S}4)^2-m\mathfrak{S}8\}+\\ 2(\mathfrak{S}3)^{-1}\mathfrak{S}5\mathfrak{S}6-\\ (\mathfrak{S}3)^{-2}\mathfrak{S}4(\mathfrak{S}5)^2.\end{gathered}$$

Now by means of Newton's theorem,

$$\mathfrak{S}\mu+A\mathfrak{S}(\mu-1)+B\mathfrak{S}(\mu-2)+C\mathfrak{S}(\mu-3)+\ldots=0,$$

we shall find, on taking A, B, E, F, G severally equal to zero, that

$$\mathfrak{S}5=0,\quad \mathfrak{S}6=3^{-1}(\mathfrak{S}3)^2,\quad \mathfrak{S}7=(3^{-1}+4^{-1})\mathfrak{S}3\mathfrak{S}4,$$
$$\mathfrak{S}8=4^{-1}(\mathfrak{S}4)^2.$$

In the case, therefore, of the biquadratic equation,

$$x^4+Cx+D=0,$$

---

the nth degree with respect to ν quantities u_1, u_2, u_3, . . u_ν, designated by $\mathfrak{f}(u_1, u_2, \ldots u_\nu)^n$, we put $u_k=a_kv+b_kw$, we shall have

$$\begin{aligned}&\mathfrak{f}(u_1, u_2, \ldots u_\nu)^n=\\ &\mathfrak{f}(a_1, a_2, \ldots u_\nu)^n v^n+\\ &n\mathfrak{f}(a_1, a_2, \ldots a_\nu)^{n-1}(b_1, b_2, \ldots b_\nu)^1 v^{n-1}w+\\ \frac{n(n-1)}{1\,.\,2}&\mathfrak{f}(a_1, a_2, \ldots a_\nu)^{n-2}(b_1, b_2, \ldots b_\nu)^2 v^{n-2}w^2+\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &\mathfrak{f}(b_1, b_2, \ldots b_\nu)^n w^n;\end{aligned}$$

a theorem which will often be found useful.

the expressions for α, β, γ will become

$$\alpha = -3^{-1} \times 4^{-1}(\mathfrak{S}3)^2 - (\mathfrak{S}3)^{-2}(\mathfrak{S}4)^3,$$
$$\beta = (3^{-1}+4^{-1})(1-1)\mathfrak{S}3\mathfrak{S}4,$$
$$\gamma = 4^{-1}(1-1)(\mathfrak{S}4)^2.$$

The equation between S and T will thus contract into

$$\alpha S^2 = 0,$$

which will give

$$S = 0;$$

except for that particular relation between C and D which causes α to vanish*.

Again, from the equations for expressing P and R in terms of S and T, we obtain, when $S=0$,

$$P = DT, \quad R = 0.$$

Further, if we subtract $y = P + Qx + Rx^2 + Sx^3 + Tx^4$ from $0 = T(D + Cx + x^4)$ and make the requisite substitutions, we shall find

$$(TC - Q)x + y = 0;$$

and therefore

$$(TC - Q)(x_1 + \kappa x_2) + y_1 + \kappa y_2 = 0:$$

whence there will arise, if $\kappa = -\frac{y_1}{y_2}$,

$$(TC - Q)\left(x_1 - \frac{y_1}{y_2}x_2\right) = 0.$$

Accordingly, except when we are unable to select x_1 and x_2 so that the second factor in this equation shall be different from zero†, we must have

$$TC - Q = 0;$$

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[*] The actual expression for α in terms of C and D is

$$\alpha = -\frac{3}{4}C^2 + \frac{4^3}{3^2}\frac{D^3}{C^2};$$

as will be manifest on observing that in the present instance

$$\mathfrak{S}3 = -3C, \quad \mathfrak{S}4 = -4D.$$

[†] The only case in which such a selection is not possible is that of $C=0$. For, there must subsist

$$\frac{y_1}{y_2} = (\sqrt{-1})^k;$$

since y_2, y_3, y_4, which are supposed to correspond to x_2, x_3, x_4 respectively, may, in an undetermined order, the fixing of which would define k, be represented by $\sqrt{-1}y_1$, $(\sqrt{-1})^2y_1$, $(\sqrt{-1})^3y_1$.

and consequently

$$y=0.$$

We conclude, then, that unless either

$$\frac{3}{4}C^2-\frac{4^3}{3^2}\frac{D^2}{C^2}=0,$$

or

$$C=0,$$

the equation

$$y=\mathrm{P}+\mathrm{Q}x+\mathrm{R}x^2+\mathrm{S}x^3+\mathrm{T}x^4$$

will, in the case under discussion, be identical with

$$0=\mathrm{T}(D+Cx+0x^2+0x^3+x^4).$$

Let us see what occurs in passing through the collapse.

MODE OF SOLUTION WHEN $\lambda=3$.

40. Taking for the equations of condition

$$\left.\begin{aligned}&\mathfrak{S}.(0\mathrm{P}+1\mathrm{Q}+2\mathrm{R}+3\mathrm{S})^1=0,\\&\mathfrak{S}.(0\mathrm{P}+1\mathrm{Q}+2\mathrm{R}+3\mathrm{S})^2=0,\\&\mathfrak{S}.(0\mathrm{P}+1\mathrm{Q}+2\mathrm{R}+3\mathrm{S})^3=0;\end{aligned}\right\}\quad .\quad .\quad .\quad .\quad (e)$$

and expanding them, as before, according to the descending powers of Q, let

$$\mathfrak{S}1.(0\mathrm{P}+2\mathrm{R}+3\mathrm{S})=\text{מ}^{*};$$

then, on eliminating P and R, the second of the equations (e), which is such as to become $\alpha\mathrm{S}^2=0$ when $\text{מ}=0$†, must, when מ is indeterminate, present itself in the form

$$2\text{מ}\mathrm{Q}+\alpha\mathrm{S}^2+2\mathrm{b}\text{מ}\mathrm{S}+\mathrm{c}\text{מ}^2=0;\quad .\quad .\quad .\quad .\quad (e'_2)$$

in which b, c are rational functions of $C, D, \ldots V$, the coefficients of the equation in x.

It may be proved very readily, that both b and c will in general be different from zero.

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* מ is the Hebrew letter Mēm.

† As T is here equal to zero, the equation $\alpha\mathrm{S}^2+2\beta\mathrm{ST}+\gamma\mathrm{T}^2=0$ is reducible to $\alpha\mathrm{S}^2=0$ irrespectively of β and γ, which, when $m>4$, are in general non-evanescent.

In effect, if we observe that

$$\mathfrak{S}\,.\,(0P+1Q+2R+3S)^2=$$
$$0Q^2+2\mathfrak{D}Q+\mathfrak{S}\,.\,(0P+2R+3S)^2,$$

we shall find, if $P=pS+p_{,}\mathfrak{D}$, $R=rS+r_{,}\mathfrak{D}$,

$$\mathrm{b}=\mathfrak{S}\,.\,(0p+2r+3s)(0p_{,}+2r_{,}),$$
$$\mathrm{c}=\mathfrak{S}\,.\,(0p_{,}+2r_{,})^2;$$

the new quantities $p_{,}$, $r_{,}$ being respectively equal to 0, $-\frac{1}{\mathfrak{S}3}$.

Whence it appears, that, unless certain assignable relations exist among the coefficients of the equation in x, both b and c will be composed of finite non-evanescent factors.

Reverting to the equation (e'_2), let us now take

$$Q=\mu\mathfrak{D}+\sigma S+q,$$

and we shall have, on eliminating Q,

$$2\mathfrak{D}Q+\alpha S^2+2\mathrm{b}'\mathfrak{D}S+\mathrm{c}'\mathfrak{D}^2=0\,; \quad . \quad . \quad . \quad (e''_2)$$

an equation which is of the same form, indeed, as the preceding one designated by (e'_2), but which involves two additional indeterminate quantities μ and σ. In effect,

$$\mathrm{b}'=\mathrm{b}+\sigma, \quad \mathrm{c}'=\mathrm{c}+2\mu.$$

We see, then, that the equation (e''_2) may be reduced to the binómial form,

$$2\mathfrak{D}q+\alpha S^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e'''_2)$$

by assigning such values to σ and μ as will make b′ and c′ vanish.

Again, if we eliminate P, R, and Q from the third of the equations (e), we shall arrive at a homogeneous equation of the third degree relatively to $\mathfrak{D}$, q, and S,

$$F(\mathfrak{D},\,q,\,S)^3=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e'_3$$

where F is indicative of a rational and integral function.

It only therefore remains to consider under what circumstances we can satisfy the simultaneous equations (e'''_2), (e'_3).

Of the applicability of the method in the case of m = 4. *Decomposition of the final equation of the sixth degree considered. Nature of the coefficients of that equation.*

41. Now if, assuming $S=1$, we designate by

$$\mathfrak{D}^6+B_1\mathfrak{D}^5+B_2\mathfrak{D}^4+\ldots+B_6=0,$$

the final equation in ט, and by

$$q^6+C_1q^5+C_2q^4+\ldots+C_6=0,$$

that in q; we may without difficulty perceive that, when $m=4$,

$$B_n=C_n;$$

n being equal to any number in the series 1, 2, 3, 4, 5, 6. For let

$$q^{-1}=q, \quad -\frac{2}{\alpha}\text{ט}=\dot{\text{ט}};$$

then, since by the equation (e'''_2),

$$\dot{q}=\dot{\text{ט}},$$

we must have

$$\dot{q}_n=\dot{\text{ט}}_n:$$

there being nothing in the nature of the process to individuate from the rest any one of the six values of which, as we know beforehand, Q is susceptible; or any one of the six sets of values of P, R, S, from which springs ט; so as to fix q and ט*.

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* In perfect accordance with this, if we take, as Lagrange has done,

$$x_1^3+Rx_1^2+Qx_1+P=+y_1,$$
$$x_2^3+Rx_2^2+Qx_2+P=-y_1,$$
$$x_3^3+Rx_3^2+Qx_3+P=+\sqrt{-1}\,y_1,$$
$$x_4^3+Rx_4^2+Qx_4+P=-\sqrt{-1}\,y_1,$$

and combine these equations so as to eliminate y_1 and P, there will result

$$\begin{aligned}&(x_1+x_2-x_3-x_4)Q+\\&(x_1^2+x_2^2-x_3^2-x_4^2)R+\\&\ x_1^3+x_2^3-x_3^3-x_4^3=0,\end{aligned}$$

$$\begin{aligned}&[x_1-x_2+(x_3-x_4)\sqrt{-1}]Q+\\&[x_1^2-x_2^2+(x_3^2-x_4^2)\sqrt{-1}]R+\\&\ x_1^3-x_2^3+(x_3^3-x_4^3)\sqrt{-1}=0,\end{aligned}$$

which will give six-valued forms for Q and R in terms of x_1, x_2, x_3, x_4. And we shall seemingly arrive at a six-valued expression capable of being made equivalent either to $\dot{q}$ or $\dot{\text{ט}}$. But we must not conclude, whatever may be the result when $m=4$, that when $m>4$, if we endeavour to obtain a function of $x_1, x_2, \ldots x_m$, which shall represent indifferently $\dot{q}$ or $\dot{\text{ט}}$, no equation of condition will steal in to interrupt the process.

It follows, therefore, that the first members of the equations

$$\dot{q}^6+\frac{C_5}{C_6}\dot{q}^5+\frac{C_4}{C_6}\dot{q}^4+\ldots+\frac{C_0}{C_6}=0,$$

$$\dot{\text{ט}}^6-\frac{2B_1}{\alpha}\dot{\text{ט}}^5+\frac{2^2B_2}{\alpha^2}\dot{\text{ט}}^4-\ldots+\frac{2^6B_6}{\alpha^6}=0$$

must be identical. Now in order that the corresponding coefficients in these equations may be equal, they must all of them be comprised in the expression

$$\frac{C_\nu}{C_6}=\left(\frac{-2}{\alpha}\right)^{6-\nu}B_{6-\nu},$$

on taking ν successively equal to 5, 4, 3, 2, 1, 0. But ט and q are involved symmetrically in the equation (e'''_2). There must accordingly exist a parallel system of conditions derivable from the equation

$$\frac{B_\nu}{B_6}=\left(\frac{-2}{\alpha}\right)^{6-\nu}C_{6-\nu}.$$

Hence it is manifest that the roots of the final equation in ט or q will be expressible by

$$q_1,\ q_2,\ q_3,\ \text{ט}_1,\ \text{ט}_2,\ \text{ט}_3,$$

while

$$q_n\text{ט}_n=-\tfrac{1}{2}\alpha.$$

P, Q, R may consequently all of them be determined by resolving equations of the first, second, and third degrees.

42. It will be interesting, however, to examine the way in which the decomposition of the equation in ט actually takes place.

Observing that

$$q_1\text{ט}_1=-\tfrac{1}{2}\alpha,$$

it is clear that if we put

$$q_1+\text{ט}_1=u_1,$$

we shall be conducted to an equation of the second degree in ט,

$$\text{ט}^2-u_1\text{ט}-\tfrac{1}{2}\alpha=0,$$

which must be a divisor of

$$\text{ט}^6+B_1\text{ט}^5+B_2\text{ט}^4+\ldots+B_6=0.$$

With respect to u_1, it is a root of a cubic equation which I shall designate by

$$u^3+D_1\,u^2+D_2\,u+D_3=0,$$

D_1, D_2, D_3 being expressible as rational functions of $B_1, B_2, \ldots B_6$, and therefore ultimately of $C, D, \ldots V$, the coefficients of the equation in x.

If, in effect, we put

$$u_n=q_n+\mathfrak{D}_n,$$

and thence (41),

$$u_n=\mathfrak{D}_n-\frac{\alpha}{2\mathfrak{D}_n},$$

the six values of which u, a rational function of $\mathfrak{D}$, is susceptible, will, since we are permitted to take $\mathfrak{D}_4$, $\mathfrak{D}_5$, $\mathfrak{D}_6$ equal to q_1, q_2, q_3 respectively, admit of being represented by

$$\mathfrak{D}_1-\frac{\alpha}{2\mathfrak{D}_1},\quad \mathfrak{D}_2-\frac{\alpha}{2\mathfrak{D}_2},\quad \mathfrak{D}_3-\frac{\alpha}{2\mathfrak{D}_3},$$

$$-\frac{\alpha}{2\mathfrak{D}_1}+\mathfrak{D}_1,\quad -\frac{\alpha}{2\mathfrak{D}_2}+\mathfrak{D}_2,\quad -\frac{\alpha}{2\mathfrak{D}_3}+\mathfrak{D}_3.$$

Wherefore the equation in u, although it will rise to the sixth degree, will be of the form

$$(u^3+D_1\,u^2+D_2\,u+D_3)^2=0,$$

composed of two equal cubic factors.

Hence it is manifest that the equation $\mathfrak{D}^6+B_1\mathfrak{D}^5+\ldots=0$ is capable of being split into three quadratic equations, in each of which the absolute term is equal to $-\frac{1}{2}\alpha$, and the coefficient of the first power of $\mathfrak{D}$ is a distinct root, with its sign changed, of a cubic equation, the coefficients of which can be expressed as known rational functions of those of the original equation in x.

43. I have not as yet shown how to deduce from the expressions in 41 the relations which exist among $B_1, B_2, \ldots B_6$.

Setting out with

$$C_n=B_n,$$

we obtain at once

$$\frac{B_n}{B_6}=\left(\frac{-2}{\alpha}\right)^{6-n}B_{6-n}\,;$$

from which will arise, on writing $6-n$ for n,

$$\frac{B_{6-n}}{B_6}=\left(\frac{-2}{\alpha}\right)^{n}B_n.$$

Again, if we multiply the corresponding members of these equations together, the result will be divisible by $B_n\,B_{6-n}$; so that we shall have

$$\left(\frac{1}{B_6}\right)^2=\left(\frac{-2}{\alpha}\right)^6,$$

and therefore

$$\frac{1}{B_6}={}_2\rho\left(\frac{-2}{\alpha}\right)^3;$$

${}_2\rho$ being a root, not yet fixed, of the equation ${}_2\rho^2-1=0$.

Eliminating B_6 from the fundamental equation in B_n, we shall thence find

$$B_n={}_2\rho^{-1}\left(\frac{-2}{\alpha}\right)^{3-n}B_{6-n},$$

if

$${}_2\rho^{-1}=1;$$

as will be manifest on taking $n=3$.

We must accordingly have

$$B_4=\left(\frac{-2}{\alpha}\right)^{-1}B_2,$$

$$B_5=\left(\frac{-2}{\alpha}\right)^{-2}B_1,$$

$$B_6=\left(\frac{-2}{\alpha}\right)^{-3}B_0;$$

B_0 being equal to 1.

44. But are these equations homogeneous with respect to the roots of

$$x^4+Cx+D=0\,?$$

If they are not, the method just given is not directly applicable to that equation, the coefficients of which, C, D, are supposed to continue arbitrary. Taking, however,

$$px=\dot{x},$$

and subjecting the equation

$$\dot{x}^4+Cp^3\dot{x}+Dp^4=0$$

to the method in question, we shall find, as might easily be shown,

$$B_4=\left(\frac{-2}{\alpha}\right)^{-1}p^{-2}B_2,$$

$$B_5=\left(\frac{-2}{\alpha}\right)^{-2}p^{-4}B_1,$$

$$B_6=\left(\frac{-2}{\alpha}\right)^{-3}p^{-6}B_0;$$

in which the homogeneity is undisturbed.

Still it may be asked, Can this system of equations be satisfied without inducing any relation between C and D?

Deferring for a time the discussion of these questions, I proceed to explain another method which may without difficulty be seen to be applicable to every biquadratic equation.

Isolated method for effecting the reduction of any equation of the fourth degree to the binomial form. The final equation. Change which takes place in the structure of the expressions at the collapse.

45. Let, as before,

$$x^4+Ax^3+Bx^2+Cx+D=0$$

denote the equation in x. Then if in the subsidiary equation

$$x^3+Rx^2+Qx+P=y$$

we write

$$x_1, \quad x_2, \quad x_3, \quad x_4$$

successively for x, and

$$+y_1, \quad -y_1, \quad +y_1\sqrt{-1}, \quad -y_1\sqrt{-1}$$

for y, and combine the resulting equations so as to eliminate y_1 and P, we shall, as has been observed*, be conducted to the equations

$$\left.\begin{array}{l}(x_1+x_2-x_3-x_4)Q+\\(x_1^2+x_2^2-x_3^2-x_4^2)R+\\x_1^3+x_2^3-x_3^3-x_4^3=0,\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (e_1)$$

$$\left.\begin{array}{l}[x_1-x_2+(x_3-x_4)\sqrt{-1}]Q+\\{}[x_1^2-x_2^2+(x_3^2-x_4^2)\sqrt{-1}]R+\\x_1^3-x_2^3+(x_3^3-x_4^3)\sqrt{-1}=0.\end{array}\right\} . \quad . \quad . \quad (e_2)$$

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* See note, p. 29.

We might therefore, following in the footsteps of Lagrange, express Q and R as non-symmetrical functions of the roots x_1, x_2, x_3, x_4; and thence by means of the theory of combinations determine the nature of the equations on which those functions depend. But without the aid of a theory, the elucidation of which in its application to equations is reserved for the next chapter, we shall be able to derive from Tschirnhausen's method all that is requisite for our present purpose.

46. It is clear that we may assign to x_1^2, x_2^2, x_3^2, x_4^2 and to x_1^3, x_2^3, x_3^3, x_4^3 systems of expressions of the same type as that in 25 for x_1, x_2, x_3, x_4.

In effect, when n is any integer, we can make

$$\left.\begin{aligned} x_1^n &= a_n + b_n\sqrt{\mathrm{I}} + (c_n + d_n\sqrt{\mathrm{I}})\sqrt{\Theta + \sqrt{\mathrm{I}}}, \\ x_2^n &= a_n + b_n\sqrt{\mathrm{I}} - (c_n + d_n\sqrt{\mathrm{I}})\sqrt{\Theta + \sqrt{\mathrm{I}}}, \end{aligned}\right\}$$

$$\left.\begin{aligned} x_3^n &= a_n - b_n\sqrt{\mathrm{I}} + (c_n - d_n\sqrt{\mathrm{I}})\sqrt{\Theta - \sqrt{\mathrm{I}}}, \\ x_4^n &= a_n - b_n\sqrt{\mathrm{I}} - (c_n - d_n\sqrt{\mathrm{I}})\sqrt{\Theta - \sqrt{\mathrm{I}}}, \end{aligned}\right\}$$

a_n, b_n, c_n, d_n being integral functions of a, b, c, d, Θ, I.

Nothing therefore is easier than to see that the coefficients in the equation (e_1) will be included under the form

$$4b_n\sqrt{\mathrm{I}};$$

and those in (e_2) under the form

$$2\{(c_n + d_n\sqrt{\mathrm{I}})\sqrt{\Theta + \sqrt{\mathrm{I}}} + (c_n - d_n\sqrt{\mathrm{I}})\sqrt{\Theta - \sqrt{\mathrm{I}}}\,\sqrt{-1}\}.$$

Hence if we denote

$$(c_n + d_n\sqrt{\mathrm{I}})\sqrt{\Theta + \sqrt{\mathrm{I}}}, \quad (c_n - d_n\sqrt{\mathrm{I}})\sqrt{\Theta - \sqrt{\mathrm{I}}},$$

by

$$\mathrm{K}_n, \qquad \mathrm{L}_n,$$

respectively, the equations between Q and R will be reducible to

$$\left.\begin{aligned} &b\mathrm{Q} + b_2\mathrm{R} + b_3 = 0, \\ &(\mathrm{K} + \mathrm{L}\sqrt{-1})\mathrm{Q} + (\mathrm{K}_2 + \mathrm{L}_2\sqrt{-1})\mathrm{R} + \mathrm{K}_3 + \mathrm{L}_3\sqrt{-1} = 0, \end{aligned}\right\} \ldots (\epsilon)$$

in which b, b_2, b_3, K, K_2, K_3, L, L_2, L_3 are known functions of the coefficients A, B, C, D of the original equation in x.

47. In the system from which these equations between Q and

R have been derived,

$$+y_1\sqrt{-1},\qquad -y_1\sqrt{-1}$$

correspond with

$$x_3,\qquad\qquad x_4,$$

in the order of their occurrence; but a system differing from the preceding one merely by the interchange of $+\sqrt{-1}$ and $-\sqrt{-1}$ or, which is the same thing, by that of x_3 and x_4, would give

$$\left.\begin{array}{l} bQ+b_2R+b_3=0,\\ (K-L\sqrt{-1})Q+(K_2-L_2\sqrt{-1})R+K_3-L_3\sqrt{-1}=0.\end{array}\right\}\ldots(\epsilon')$$

48. To remind us of the separate existence of the two systems from which spring (ϵ) and (ϵ'), I shall annex $+\sqrt{-1}$ as an index to the functions designated by Q and R in the first system, and $-\sqrt{-1}$ to those in the second.

Thus, if we confine our attention to R, we shall have

$$R_{+\sqrt{-1}}=-\frac{b_3(K+L\sqrt{-1})-b(K_3+L_3\sqrt{-1})}{b_2(K+L\sqrt{-1})-b(K_2+L_2\sqrt{-1})},$$

$$R_{-\sqrt{-1}}=-\frac{b_3(K-L\sqrt{-1})-b(K_3-L_3\sqrt{-1})}{b_2(K-L\sqrt{-1})-b(K_2-L_2\sqrt{-1})};$$

the expresion for $R_{-\sqrt{-1}}$ being derivable from that for $R_{+\sqrt{-1}}$ by writing throughout each part $-\sqrt{-1}$ for $+\sqrt{-1}$.

49. It may now be seen that the coefficients of the quadratic equation

$$(R-R_{+\sqrt{-1}})(R-R_{-\sqrt{-1}})=0$$

are rational functions of a, b, c, d, Θ, I.

If, in effect, we bring the numerators and denominators of the expressions for $R_{+\sqrt{-1}}$ and $R_{-\sqrt{-1}}$ to the form $\alpha+\beta\sqrt{-1}$ by assuming

$$t_0=b_3K-bK_3,\quad t_1=b_3L-bL_3,$$
$$u_0=b_2K-bK_2,\quad u_1=b_2L-bL_2,$$

there will manifestly result

$$-(R_{+\sqrt{-1}}+R_{-\sqrt{-1}})=\frac{2(t_0u_0+t_1u_1)}{u_0^2+u_1^2},$$

$$R_{+\sqrt{-1}}\times R_{-\sqrt{-1}}=\frac{t_0^2+t_1^2}{u_0^2+u_1^2}.$$

50. Again, as we can put

$$t_0=\dot{t}_0\sqrt{\Theta+\sqrt{\mathrm{I}}},\quad t_1=\dot{t}_1\sqrt{\Theta-\sqrt{\mathrm{I}}},$$
$$u_0=\dot{u}_0\sqrt{\Theta+\sqrt{\mathrm{I}}},\quad u_1=\dot{u}_1\sqrt{\Theta-\sqrt{\mathrm{I}}},$$

where $\dot{t}_0$, $\dot{t}_1$, $\dot{u}_0$, $\dot{u}_1$ are free from compound radicals, it is clear that the functions

$$\frac{2(t_0u_0+t_1u_1)}{u_0^2+u_1^2},$$
$$\frac{t_0^2+t_1^2}{u_0^2+u_1^2}$$

may be expressed by

$$\frac{2[\dot{t}_0\dot{u}_0\sqrt{(\Theta+\sqrt{\mathrm{I}})^2}+\dot{t}_1\dot{u}_1\sqrt{(\Theta-\sqrt{\mathrm{I}})^2}]}{\dot{u}_0^2\sqrt{(\Theta+\sqrt{\mathrm{I}})^2}+\dot{u}_1^2\sqrt{(\Theta-\sqrt{\mathrm{I}})^2}},$$
$$\frac{\dot{t}_0^2\sqrt{(\Theta+\sqrt{\mathrm{I}})^2}+\dot{t}_1^2\sqrt{(\Theta-\sqrt{\mathrm{I}})^2}}{\dot{u}_0^2\sqrt{(\Theta+\sqrt{\mathrm{I}})^2}+\dot{u}_1^2\sqrt{(\Theta-\sqrt{\mathrm{I}})^2}};$$

in which the numerators and denominators severally belong to the class

$$v+w\sqrt{\mathrm{I}}+v-w\sqrt{\mathrm{I}};$$

v and w being rational functions of a, b, c, d, Θ, I: as will be evident on considering that $+\sqrt{\mathrm{I}}$ and $-\sqrt{\mathrm{I}}$ enter symmetrically into the calculus.

51. We are thus brought to an equation of the form

$$\mathrm{R}^2+\zeta\mathrm{R}+\eta=0,$$

where ζ and η are rational functions of a, b, c, d, Θ, I.

52. Lastly, since a, b, c, d, Θ, I all of them in general admit of being expressed as determinate rational functions of $\dot{\mathrm{Q}}$ (the point over the letter serving to distinguish it from the Q in the present series for y), we see that if we denote by $\dot{\mathrm{Q}}_1$, $\dot{\mathrm{Q}}_2$, $\dot{\mathrm{Q}}_3$ the three roots of the cubic equation in $\dot{\mathrm{Q}}$, and by ζ_1, ζ_2, ζ_3, η_1, η_2, η_3 the corresponding values of ζ and η, the equation of the sixth degree in R will be decomposable into

$$(\mathrm{R}^2+\zeta_1\mathrm{R}+\eta_1)\times$$
$$(\mathrm{R}^2+\zeta_2\mathrm{R}+\eta_2)\times$$
$$(\mathrm{R}^2+\zeta_3\mathrm{R}+\eta_3)=0;$$

which differs essentially in structure from the final equation in 35.

53. The problem which I intend now to propose for discussion is one the solution of which would embrace that of equations of the fifth degree. It is

TO REDUCE THE EQUATION

$$x^m + Dx^{m-4} + Ex^{m-5} + \ldots + V = 0$$

TO THE FORM

$$y^m + B'y^{m-2} + kB'^2y^{m-4} + E'y^{m-5} + \ldots + V' = 0\,;$$

k BEING ANY NUMERICAL CONSTANT*.

Writing for greater symmetry Q_0, Q_1, Q_2, .. Q_λ, instead of P, Q, R, .. L respectively in

$$y = \mathrm{P} + \mathrm{Q}x + \mathrm{R}x^2 + \ldots + \mathrm{L}x^\lambda,$$

we immediately perceive that the equations of condition which must be satisfied in order that we may have

$$A' = 0, \quad C' = 0, \quad D' - kB'^2 = 0,$$

will be

$$\mathfrak{S} \cdot (0\mathrm{Q}_0 + 1\mathrm{Q}_1 + 2\mathrm{Q}_2 + \ldots + \lambda \mathrm{Q}_\lambda)^1 = 0,$$

$$\mathfrak{S} \cdot (0\mathrm{Q}_0 + 1\mathrm{Q}_1 + 2\mathrm{Q}_2 + \ldots + \lambda \mathrm{Q}_\lambda)^3 = 0,$$

$$\mathfrak{S} \cdot (0\mathrm{Q}_0 + 1\mathrm{Q}_1 + 2\mathrm{Q}_2 + \ldots + \lambda \mathrm{Q}_\lambda)^4 - 1 \,.\, 2 \,.\, 3k[\mathfrak{S} \cdot (0\mathrm{Q}_0 + 1\mathrm{Q}_1 + 2\mathrm{Q}_2 + \ldots + \lambda \mathrm{Q}_\lambda)^2]^2 = 0.$$

Now if we expand the first two of these according to the de-

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* When $m=5$, $k=\frac{1}{5}$, the equation in y will become

$$y^5 + B'y^3 + \tfrac{1}{5}B'^2y + E' = 0,$$

the roots of which may, as De Moivre has shown, be represented thus:

$$\sqrt[5]{\mathrm{K} + \sqrt{\Lambda}} + \sqrt[5]{\mathrm{K} - \sqrt{\Lambda}},$$

$$\iota\sqrt[5]{\mathrm{K} + \sqrt{\Lambda}} + \iota^4\sqrt[5]{\mathrm{K} - \sqrt{\Lambda}},$$

$$\iota^2\sqrt[5]{\mathrm{K} + \sqrt{\Lambda}} + \iota^3\sqrt[5]{\mathrm{K} - \sqrt{\Lambda}},$$

$$\iota^3\sqrt[5]{\mathrm{K} + \sqrt{\Lambda}} + \iota^2\sqrt[5]{\mathrm{K} - \sqrt{\Lambda}},$$

$$\iota^4\sqrt[5]{\mathrm{K} + \sqrt{\Lambda}} + \iota\sqrt[5]{\mathrm{K} - \sqrt{\Lambda}};$$

K, Λ being defined by

$$\mathrm{K} = -\frac{E'}{2}, \quad \Lambda = \left(-\frac{E'}{2}\right)^2 + \left(\frac{B'}{5}\right)^5;$$

and 1, ι, ι^2, ι^3, ι^4 being the roots of the binomial equation

$$\rho^5 - 1 = 0.$$

scending powers of Q_1 or Q, there will result

$$\begin{aligned} &0Q+\\ &\mathfrak{S}\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^1=0 \end{aligned}$$

and

$$\begin{aligned} &0Q^3+\\ &3\mathfrak{S}1^2\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^1\,Q^2+\\ &3\mathfrak{S}1\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^2\,Q+\\ &\mathfrak{S}\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^3=0; \end{aligned}$$

$\mathfrak{S}1$ and $\mathfrak{S}1^3$ being both of them evanescent. In order, therefore, to detach Q, as before, we must be able to assign such values to $Q_0, Q_2, \ldots Q_\lambda$ as will satisfy the equations

$$\begin{aligned} &\mathfrak{S}\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^1=0,\\ &\mathfrak{S}1^2\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^1=0,\\ &\mathfrak{S}1\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^2=0,\\ &\mathfrak{S}\,.\,(0Q_0+2Q_2+\ldots+\lambda Q_\lambda)^3=0; \end{aligned}$$

where the product of the numbers which mark the dimensions relatively to $Q_0, Q_2, \ldots Q_\lambda$ is $1\,.\,1\,.\,2\,.\,3$, and is therefore greater than 4, the highest number which is admissible in our present inquiry. The difficulty may, however, be overcome. For if, assuming

$$\begin{aligned} Q_0&=a_0M+b_0,\\ Q_2&=a_2M+b_2,\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ Q_\lambda&=a_\lambda M+b_\lambda, \end{aligned}$$

we express the first three of the equations in $Q_0, Q_2, \ldots Q_\lambda$ as functions of $a_0, a_2, \ldots a_\lambda$, $b_0, b_2, \ldots b_\lambda$, and M, we shall have

$$\begin{aligned} &\mathfrak{S}\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^1\,M+\\ &\mathfrak{S}\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^1=0,\\[1ex] &\mathfrak{S}1^2\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^1\,M+\\ &\mathfrak{S}1^2\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^1=0,\\[1ex] &\mathfrak{S}1\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^2\,M^2+\\ 2&\mathfrak{S}1\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^1(0b_0+2b_2+\ldots+\lambda b_\lambda)^1\,M+\\ &\mathfrak{S}1\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^2=0. \end{aligned}$$

Accordingly if, as is permitted, we give such values to $a_0, a_2, \ldots a_\lambda$,

$b_0, b_2, \ldots b_\lambda$, that they may admit of being arranged in the two divisions or groups,

I.

$$\mathfrak{S}\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^1=0,$$
$$\mathfrak{S}1^2\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^1=0,$$
$$\mathfrak{S}1\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^2=0,$$

II.

$$\mathfrak{S}\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^1=0,$$
$$\mathfrak{S}1^2\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^1=0,$$
$$\mathfrak{S}1\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^1(0b_0+2b_2+\ldots+\lambda b_\lambda)^1=0,$$
$$\mathfrak{S}1\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^2=0,$$

the final equation for $a_0, a_2, \ldots a_\lambda$ being of $1\,.\,1\,.\,2$ dimensions, and that for $b_0, b_2, \ldots b_\lambda$ of $1\,.\,1\,.\,1\,.\,2$; then M will be detached from the three equations in question, and will consequently be wholly free on entering

$$\mathfrak{S}\,.\,(0\mathrm{Q}_0+2\mathrm{Q}_2+\ldots+\lambda \mathrm{Q}_\lambda)^3=0,$$

that is to say,

III.

$$\begin{aligned}&\mathfrak{S}\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^3\,\mathrm{M}^3+\\ 3&\mathfrak{S}\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)^2(0b_0+2b_2+\ldots+\lambda b_\lambda)\,\mathrm{M}^2+\\ 3&\mathfrak{S}\,.\,(0a_0+2a_2+\ldots+\lambda a_\lambda)(0b_0+2b_2+\ldots+\lambda b_\lambda)^2\,\mathrm{M}+\\ &\mathfrak{S}\,.\,(0b_0+2b_2+\ldots+\lambda b_\lambda)^3=0,\end{aligned}$$

the last of the equations which do not involve Q.

Thus we see how, without resolving any equation of a higher degree than the third, we shall have

$$\begin{aligned}A'&=0\mathrm{Q}+0\mathrm{M}+0,\\ C'&=0\mathrm{Q}^3+(0\mathrm{M}+0)\mathrm{Q}^2\\ &\qquad+(0\mathrm{M}^2+0\mathrm{M}+0)\mathrm{Q}+0.\end{aligned}$$

And as Q may now be determined so as to satisfy the biquadratic equation

IV.

$$\begin{aligned}&\mathfrak{S}\,.\,(0\mathrm{Q}_0+1\mathrm{Q}+2\mathrm{Q}_2+\ldots+\lambda\mathrm{Q}_\lambda)^4-\\ 1\,.\,2\,.\,3k[&\mathfrak{S}\,.\,(0\mathrm{Q}_0+1\mathrm{Q}+2\mathrm{Q}_2+\ldots+\lambda\mathrm{Q}_\lambda)^2]^2=0,\end{aligned}$$

and therefore

$$D'-kB'^2=0\,;$$

it is visible that the transformation we are considering may be effected by means of equations of the first four degrees. But it remains to assign suitable values to λ.

NON-OCCURRENCE, EXCEPT IN PARTICULAR CASES, OF THE FORM $\frac{\mathbf{G}}{(1-1)\mathbf{H}}$ WHEN $\lambda=6$.

54. Returning to the first group in which are $a_0, a_2, \ldots a_\lambda$, let us assume

$$a_5=0, \quad a_6=a_\lambda=0,$$

we shall then have four indeterminate quantities for satisfying the three homogeneous equations there involved.

Now on combining the first and second of these equations, and supposing for the moment that

$$a_0=\frac{\mathrm{N}_0}{\mathbf{D}}, \quad a_2=\frac{\mathrm{N}_2}{\mathbf{D}},$$

we may perceive that N_0 and N_2 will both of them belong to the class

$$\Phi(a_3, a_4, \mathfrak{S}1^24, \mathfrak{S}1^23, \ldots)\,;$$

Φ being expressive of a rational and whole function.

As for **D**, it can be made to depend on the equation

$$\mathbf{D}=\mathfrak{S}0\mathfrak{S}1^22-\mathfrak{S}2\mathfrak{S}1^20\,;$$

the order of the terms fixing that in N_0 and in N_2. Hence **D** is in general different from zero.

If therefore we reflect that $\mathfrak{S}\alpha\beta\gamma=\mathfrak{S}\alpha\mathfrak{S}\beta\mathfrak{S}\gamma-\mathfrak{S}\alpha\mathfrak{S}(\beta+\gamma)-\mathfrak{S}\beta\mathfrak{S}(\alpha+\gamma)-\mathfrak{S}\gamma\mathfrak{S}(\alpha+\beta)+2\mathfrak{S}(\alpha+\beta+\gamma)$, we shall see without difficulty that the final equation of the group will fall under the form

$$2\{2a_3\mathfrak{S}8+a_4\mathfrak{S}9\}a_4+$$
$$F\left(\frac{1}{\mathbf{D}}, a_3, a_4, \mathfrak{S}7, \mathfrak{S}6, \ldots\right)=0,$$

and will consequently, except in particular cases, contain at least two terms with non-evanescent coefficients.

In effect, the coefficient of a_4^2 being of the form

$$2\mathfrak{S}9+g,$$

where g is composed of symmetrical functions of lower dimensions than $\mathfrak{S}9$, and that of a_3a_4 of the form

$$2\times 2\mathfrak{S}8+h,$$

where h is, in like manner, composed of symmetrical functions of lower dimensions than $\mathfrak{S}8$, it follows from the principle of the equality of dimensions, already so often referred to, that we cannot have

$$2\mathfrak{S}9+g=0, \quad 2\times 2\mathfrak{S}8+h=0,$$

without inducing relations among $D, E, \ldots V$ when $m > 9$[*].

---

[*] Generally, in order that there may subsist the equation

$$Y=Z,$$

Y and Z must be of the same dimensions. Or using the character $\mathfrak{D}$ for *dimensions*, we must have

I. $$\mathfrak{D}Y=\mathfrak{D}Z.$$

Continuing thus to embody in a calculus our primary ideas connected with the notion of equality, we perceive that

$$\text{if } Z=z_1+z_2+z_3+\ldots,$$

I′. $$\mathfrak{D}Y=\mathfrak{D}z_1=\mathfrak{D}z_2=\mathfrak{D}z_3=\ldots$$

$$\text{if } Z=z_1z_2z_3\ldots,$$

I″. $$\mathfrak{D}Y=\mathfrak{D}z_1+\mathfrak{D}z_2+\mathfrak{D}z_3+\ldots$$

Hence in the general equation of the mth degree,

$$x^m+Ax^{m-1}+Bx^{m-2}+\ldots+V=0,$$

we must have (I′.)

$$\mathfrak{D}x^m=\mathfrak{D}Ax^{m-1}=\mathfrak{D}Bx^{m-2}=\ldots=\mathfrak{D}V.$$

From the first of these equations we deduce (I″.)

$$m\mathfrak{D}x=\mathfrak{D}A+(m-1)\mathfrak{D}x,$$

therefore

$$\frac{\mathfrak{D}A}{\mathfrak{D}x}=1.$$

Similarly we find,

$$\frac{\mathfrak{D}B}{\mathfrak{D}x}=2, \quad \frac{\mathfrak{D}C}{\mathfrak{D}x}=3, \ldots$$

Comparing these results with

$$\frac{\mathfrak{D}\mathfrak{S}1}{\mathfrak{D}x}=1,$$

$$\frac{\mathfrak{D}\mathfrak{S}2}{\mathfrak{D}x}=2, \quad \frac{\mathfrak{D}\mathfrak{S}3}{\mathfrak{D}x}=3, \ldots$$

and observing that all symmetrical functions of the roots are integral func-

55. Putting now $\lambda=6$ in the next group, and pursuing a method analogous to that just explained, we find that b_0, b_2, b_3 will all of them belong to the class

$$\frac{1}{'D}\,'\Phi(b_4, b_5, b_6, \mathfrak{S}11, \mathfrak{S}10, \ldots),$$

where $'\Phi$ designates a determinate function. a_0, a_2, .. a_4, entering into this group as known quantities, are not indicated in the expression.

We see too that the final equation will here be of the form

$$2\{2b_5\mathfrak{S}12+b_4\mathfrak{S}13\}b_6+$$
$$'F\left(\frac{1}{'D}, b_4, b_5, b_6, \mathfrak{S}11, \mathfrak{S}10, \ldots\right)=0.$$

---

tions of the coefficients A, B, C, .. V, we perceive why A only can enter into the expression for $\mathfrak{S}1$, A and B into that for $\mathfrak{S}2$, and so on.

Having arrived at the idea, derivable from Newton's theorem on the sums of the powers of the roots, that each symmetrical function in the series $\mathfrak{S}k$, $\mathfrak{S}(k-1)$, .. $\mathfrak{S}2$, $\mathfrak{S}1$, $\mathfrak{S}0$ will, when m is indeterminate, contain a coefficient of the original equation which does not enter into the succeeding function, we shall have little difficulty in attaining to the demonstration of the following Lemma.

No algebraical expression that is composed of symmetrical functions of the roots of the general equation of the m*th degree, and is of the class*

$${}_0\Phi\{\mathfrak{S}(k-1), \mathfrak{S}(k-2), \ldots \mathfrak{S}1, \mathfrak{S}0\}(\mathfrak{S}k)^n+$$
$${}_1\Phi\{\mathfrak{S}(k-1), \mathfrak{S}(k-2), \ldots \mathfrak{S}1, \mathfrak{S}0\}(\mathfrak{S}k)^{n-1}+$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$${}_n\Phi\{\mathfrak{S}(k-1), \mathfrak{S}(k-2), \ldots \mathfrak{S}1, \mathfrak{S}0\},$$

can become equal to zero without assuming the form

$$0(\mathfrak{S}k)^n+0(\mathfrak{S}k)^{n-1}+\ldots+0.$$

Corollary. A similar inference may be drawn respecting the function

$${}_0\Phi\{\mathfrak{S}\alpha'\beta'\gamma'\ldots, \mathfrak{S}\alpha''\beta''\gamma''\ldots, \ldots\}(\mathfrak{S}abc\ldots)^n+$$
$${}_1\Phi\{\mathfrak{S}\alpha'\beta'\gamma'\ldots, \mathfrak{S}\alpha''\beta''\gamma''\ldots, \ldots\}(\mathfrak{S}abc\ldots)^{n-1}+$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$${}_n\Phi\{\mathfrak{S}\alpha'\beta'\gamma'\ldots, \mathfrak{S}\alpha''\beta''\gamma''\ldots, \ldots\}$$

when

$$a+b+c+\ldots>\alpha^{(r)}+\beta^{(r)}+\gamma^{(r)}+\ldots;$$

as will readily appear on considering that

$$\frac{\text{Ɑ}\mathfrak{S}\alpha\beta\gamma\ldots}{\text{Ɑ}x}=\alpha+\beta+\gamma+\ldots$$

So that we can assign whatever non-evanescent value we please to b_6, and then obtain a determinate expression for b_5 if ${}'\mathfrak{D}$ is different from zero.

That ${}'\mathfrak{D}$ will not in general vanish may be shown thus. In the equation by which it may be defined,

$$
\begin{gathered}
{}'\mathfrak{D}= \\
\{\mathfrak{S}0\mathfrak{S}1^2 2-\mathfrak{S}2\mathfrak{S}1^2 0\}\mathfrak{S}1\,.\,3\,.\,(0a_0+2a_2+\,..\,+4a_4)+ \\
\{\mathfrak{S}1^2 0\mathfrak{S}3-\mathfrak{S}0\mathfrak{S}1^2 3\}\mathfrak{S}1\,.\,2\,.\,(0a_0+2a_2+\,..\,+4a_4)+ \\
\{\mathfrak{S}2\mathfrak{S}1^2 3-\mathfrak{S}3\mathfrak{S}1^2 2\}\mathfrak{S}1\,.\,0\,.\,(0a_0+2a_2+\,..\,+4a_4),
\end{gathered}
$$

the highest symmetrical function immediately involved is $\mathfrak{S}1.3.4$; we perceive, therefore, that if ${}'\mathfrak{D}=0$, we should, on eliminating a_0 and a_2, be conducted to an expression between a_3 and a_4 into which no function higher than $\mathfrak{S}8$ would enter, in opposition to what has been proved in 54, where the equation between a_3 and a_4 involves the symmetrical function $\mathfrak{S}9$.

56. Having reached this point, we may see at once that when $\lambda=6$, the series for y,

$$Q_0+Q_1x+Q_2x^2+\,..\,+Q_6x^6,$$

will in general be a determinate non-evanescent function of x.

Of the possibility of eluding the collapse which takes place when m falls below 7, I shall have occasion to speak in another place.

57. The following problem, the solution of which would involve that of equations of the sixth degree,

TO REDUCE THE EQUATION

$$x^m+Dx^{m-4}+Ex^{m-5}+Fx^{m-6}+\,..\,+V=0$$

TO THE FORM

$$y^m+B'y^{m-2}+D'y^{m-4}+F'y^{m-6}+\,..\,+V=0,$$

belongs to the same class or family as the last.

In fact, if, according to the pattern in 54, we make

$$
\begin{aligned}
y &= Q_0 + Qx + Q_2x^2 + \ldots + Q_6x^6, \\
Q_0 &= a_0M + b_0, \\
Q_2 &= a_2M + b_2, \\
Q_3 &= a_3M + b_3, \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
Q_6 &= a_6M + b_6, \\
a_5 &= 0, \\
a_6 &= 0,
\end{aligned}
$$

and determine $a_0, a_2, \ldots a_4, b_0, b_2, \ldots b_6$, M, Q, in such a manner that they may admit of being distributed into the divisions

I.

$$
\begin{aligned}
\mathfrak{S} \,.\, (0a_0 + 2a_2 + \ldots + 4a_4)^1 &= 0, \\
\mathfrak{S}1^2 \,.\, (0a_0 + 2a_2 + \ldots + 4a_4)^1 &= 0, \\
\mathfrak{S}1 \,.\, (0a_0 + 2a_2 + \ldots + 4a_4)^2 &= 0,
\end{aligned}
$$

II.

$$
\begin{aligned}
\mathfrak{S} \,.\, (0b_0 + 2b_2 + \ldots + 6b_6)^1 &= 0, \\
\mathfrak{S}1^2 \,.\, (0b_0 + 2b_2 + \ldots + 6b_6)^1 &= 0, \\
\mathfrak{S}1 \,.\, (0a_0 + 2a_2 + \ldots + 4a_4)^1(0b_0 + 2b_2 + \ldots + 6b_6)^1 &= 0, \\
\mathfrak{S}1 \,.\, (0b_0 + 2b_2 + \ldots + 6b_6)^2 &= 0,
\end{aligned}
$$

III.

$$\mathfrak{S} \,.\, (0[a_0M + b_0] + 2[a_2M + b_2] + \ldots + 6b_6)^3 = 0,$$

IV.

$$\mathfrak{S} \,.\, (0[a_0M + b_0] + 1Q + 2[a_2M + b_2] + \ldots + 6b_6)^5 = 0,$$

we shall have from the equations in I. II. III.,

$$
\begin{aligned}
A' &= 0Q + 0M + 0, \\
C' &= 0Q^3 + (0M + 0)Q^2 \\
&\qquad + (0M^2 + 0M + 0)Q + 0,
\end{aligned}
$$

and from the equation in IV.,

$$E' = 0.$$

It will be seen also precisely as before, that the series for y will in general be a determinate non-evanescent function of x.

Collapse of the series for y *in the case of* m = 6.

58. When $m=6$, we have

$$m=\lambda,$$

which necessitates a collapse. Now from what has been said in discussing the problem of taking away the second, third, and fourth terms at once from the general equation of the mth degree in the analogous case for the resolution of biquadratic equations, we are led to infer that when the equation in x here becomes

$$x^6+Dx^2+Ex+F=0,$$

the one between y and x will, in consequence of the collapse, present itself in the form

$$0=0+0x+0x^2+\,.\,.+0x^6.$$

59. Let us see how this result may be deduced from the equations themselves on which depend $a_0, a_2, \,.\,.\, a_4$, $b_0, b_2, \,.\,.\, b_6$, M, Q.

Assuming that the collapse induces

$$b_6=0,$$

we perceive that there are still five quantities, b_0, b_2, b_3, b_4, b_5, for satisfying the four homogeneous equations in II. But if, observing the similarity in form of the corresponding equations in I. and II., we take

$$\begin{aligned} b_0&=\varpi a_0, \\ b_2&=\varpi a_2+b^{\backprime}_2, \\ b^3&=\varpi a_3+b^{\backprime}_3, \\ b_4&=\varpi a_4+b^{\backprime}_4, \end{aligned}$$

it will be manifest that the equations in II. will become*

$$\begin{gathered} \mathfrak{S}\,.\,(2b^{\backprime}_2+3b^{\backprime}_3+\,.\,.+5b_5)^1=0, \\ \mathfrak{S}1^2\,.\,(2b^{\backprime}_2+3b^{\backprime}_3+\,.\,.+5b_5)^1=0, \\ \mathfrak{S}1\,.\,(0a_0+2a_2+\,.\,.+4a_4)^1(2b^{\backprime}_2+3b^{\backprime}_3+\,.\,.+5b_5)^1=0, \\ \mathfrak{S}1\,.\,(2b^{\backprime}_2+3b^{\backprime}_3+\,.\,.+5b_5)^2=0, \end{gathered}$$

---

* Thus

$$\begin{aligned} \mathfrak{S}\,.\,(0b_0+2b_2+\,.\,.+5b_5)^1=\mathfrak{S}\,.\,&(0a_0+2a_2+\,.\,.+4a_4)^1\varpi \\ &+\mathfrak{S}\,.\,(2b^{\backprime}_2+3b^{\backprime}_3+\,.\,.+5b_5)^1. \end{aligned}$$

in which the number of the quantities to be determined, $\grave{b}_2$, $\grave{b}_3$, $\grave{b}_4$, b_5, is reduced to four.

Hence, exclusively of particular cases, $\grave{b}_2$, $\grave{b}_3, \,..\, b_5$, and therefore b_0, $b_2, \,..\, b_5$ must severally vanish.

We now have

$$Q_0=a_0M, \quad Q_2=a_2M, \quad Q_3=a_3M, \quad Q_4=a_4M, \quad Q_5=0.$$

The equation in III. may accordingly be converted into

$$\mathfrak{S}\,.\,(0a_0+2a_2+\,..\,+4a_4)^3\,M^3=0.$$

Except, therefore, when $\mathfrak{S}\,.\,(0a_0+2a_2+\,..\,+4a_4)^3=0$, M will vanish, causing at the same time the evanescence of Q_0, Q_2, Q_3, Q_4.

And since, on making the requisite substitutions, the equation in IV. will become

$$\mathfrak{S}1^5\,Q^5=0,$$

Q, as well as Q_0, $Q_2, \,..\, Q_4$, must in general be equated to zero.

Adjustments to meet the collapse.

60. In the first and last of the divisions we need not make any alteration in the arrangements. But as we can no longer completely detach M, we must take the equation from which spring the third and fourth of the equations in II.,

$$2\mathfrak{S}1\,.\,(0a_0+2a_2+\,..\,+4a_4)^1(0b_0+2b_2+3b_3+\,..\,+6b_6)^1\,M$$
$$+\,\mathfrak{S}1\,.\,(0b_0+2b_2+3b_3+\,..\,+6b_6)^2=0,$$

into the third division, equating b_0 and b_6 to zero.

If, therefore, as in the analogous case in 40, we designate the coefficient of M by $2\mathfrak{D}$, we shall merely have to consider the equations in the two following divisions instead of those in II. and III.,

II′.

$$\mathfrak{S}\,.\,(2b_2+3b_3+\,..\,+5b_5)^1=0,$$
$$\mathfrak{S}1^2\,.\,(2b_2+3b_3+\,..\,+5b_5)^1=0,$$
$$\mathfrak{S}1\,.\,(0a_0+2a_2+\,..\,+4a_4)^1(2b_2+3b_3+\,..\,+5b_5)^1=\mathfrak{D},$$

III′.

$$2\mathfrak{D}M+\mathfrak{S}1\,.\,(2b_2+3b_3+\,..\,+5b_5)^2=0,$$
$$\mathfrak{S}\,.\,(0[a_0M]+2[a_2M+b_2]+\,..\,+5b_5)^3=0.$$

What the nature of the result is to which the method leads will appear in the sequel.

61. I might go on to show how, by solving a certain number of equations of the first, second, third, .. $(n-1)$th degrees, to attain to the solution of the simultaneous equations

$$
\begin{aligned}
&\mathfrak{S}\,.\,(0Q_0+1Q+2Q_2+\,..\,+\lambda Q_\lambda)^1=0,\\
&\mathfrak{S}\,.\,(0Q_0+1Q+2Q_2+\,..\,+\lambda Q_\lambda)^2=0,\\
&\mathfrak{S}\,.\,(0Q_0+1Q+2Q_2+\,..\,+\lambda Q_\lambda)^3=0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\mathfrak{S}\,.\,(0Q_0+1Q+2Q_2+\,..\,+\lambda Q_\lambda)^{n-1}=0,
\end{aligned}
$$

when a suitable value is assigned to λ. And it might be proved that till $m=\lambda$, the series for y would in general be a determinate non-evanescent function of x. But it is time for us to retrace our steps to equations of the fifth degree. The method of Tschirnhausen now merges in that of Lagrange and Vandermonde.

CHAPTER II.

INDICATIONS AFFORDED BY THE METHOD OF LAGRANGE AND VANDERMONDE OF THE POSSIBILITY OF PASSING THE LIMITS WITHIN WHICH THE RESOLUTION OF EQUATIONS HAS HITHERTO BEEN CONFINED.

62. In the preceding chapter we attempted, by a method founded on that of Tschirnhausen, to bring the general equation of the fifth degree to the form discovered by De Moivre, but were met, as in the analogous case for biquadratic equations, by the collapse of the series for y. What takes place in passing through the collapse, I have not discussed. Nor do I intend to enter upon the inquiry here. With the aid, however, of the theory of combinations, for the application of which to equations we are indebted to Lagrange and Vandermonde, we shall be able, not only to establish the truth of the proposition,

THAT ANY EQUATION OF THE FIFTH DEGREE

$$x^5 + A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5 = 0$$

CAN BE REDUCED TO THE FORM

$$y^5 + A'_2\,y^3 + \tfrac{1}{5}A'^2_2\,y + A'_5 = 0$$

BY MEANS OF A SUBSIDIARY EQUATION OF THE THIRD DEGREE WITH RESPECT TO x

$$x^3 + p_1x^2 + p_2x + p_3 = y,$$

but also to see clearly what the element is which has been omitted by later mathematicians of great eminence, who have come to the conclusion, that equations of the fifth degree cannot in general be solved algebraically.

FIRST PART OF THE DEMONSTRATION.

§ 1.

Mode of expressing, p_1, p_2, p_3 *as rational functions of* $x_1, x_2, \ldots x_5$. *The ten equations* (d.), (e.).

63. Now p_1, p_2, p_3 must be such that

$$x^3 + p_1x^2 + p_2x + p_3$$

may become a root of an equation of the form

$$y^5+A'_2 y^3+\tfrac{1}{5}A'^2_2 y+A'_5=0,$$

or that the expressions

$$\begin{array}{l} x_1^3+p_1x_1^2+p_2x_1+p_3, \\ x_2^3+p_1x_2^2+p_2x_2+p_3, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_5^3+p_1x_5^2+p_2x_5+p_3 \end{array}$$

may become the five roots of that equation. For $x_1, x_2, \ldots x_5$ enter symmetrically into the calculus, and there is consequently nothing to connect one of them rather than another with the x of the expression $x^3+p_1x^2+p_2x+p_3$.

If then we consider that the roots of the equation for y must, as De Moivre has shown*, be expressible by

$$t+u, \quad \iota t+\iota^4 u, \quad \iota^2 t+\iota^3 u, \quad \iota^3 t+\iota^2 u, \quad \iota^4 t+\iota u,$$

$\iota, \iota^2, \iota^3, \iota^4$ denoting the imaginary roots of the binomial equation $\rho^5-1=0$, we shall manifestly be conducted to a system of equations

$$\left.\begin{array}{l} x_\alpha^3+p_1x_\alpha^2+p_2x_\alpha+p_3=t+u, \\ x_\beta^3+p_1x_\beta^2+p_2x_\beta+p_3=\iota t+\iota^4 u, \\ x_\gamma^3+p_1x_\gamma^2+p_2x_\gamma+p_3=\iota^2 t+\iota^3 u, \\ x_\delta^3+p_1x_\delta^2+p_2x_\delta+p_3=\iota^3 t+\iota^2 u, \\ x_\epsilon^3+p_1x_\epsilon^2+p_2x_\epsilon+p_3=\iota^4 t+\iota u; \end{array}\right\} \quad . \quad . \quad . \quad (a)$$

in which $\alpha, \beta, \gamma, \delta, \epsilon$ represent in an undetermined or arbitrary order of succession, the five indices 1, 2, 3, 4, 5.

64. From this system there will arise, as we know from the theory of permutations, 1 . 2 . 3 . 4 . 5 systems; if for $\alpha, \beta, \gamma, \delta, \epsilon$ we substitute 1, 2, 3, 4, 5 in all the different arrangements which they can assume. But these 120 systems will be found to furnish only twelve different sets of values for p_1, p_2, p_3. Our first object will be to express p_1, p_2, p_3 as functions of $x_1, x_2, \ldots x_5$ without t and u.

65. By combining any three of the five equations of the system (a.), we see that we may eliminate t and u; and that

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* See note, p. 37.

therefore if we replace

$$x_\alpha^3+p_1x_\alpha^2+\ldots,\quad x_\beta^3+p_1x_\beta^2+\ldots,\quad x_\epsilon^3+p_1x_\epsilon^2+\ldots,$$

by $\qquad y_\alpha, \qquad y_\beta, \qquad \ldots \qquad y_\epsilon,$

respectively, we may arrive at a final equation of the first degree with respect to $y_\alpha, y_\beta, \ldots y_\epsilon$,

$$\mu_\alpha y_\alpha+\mu_\beta y_\beta+\mu_\gamma y_\gamma+\mu_\delta y_\delta+\mu_\epsilon y_\epsilon=0;$$

where $\mu_\alpha, \mu_\beta, \ldots \mu_\epsilon$ are functions of ι which have no common factor different from 1, and are such that two of them must admit of being equated to zero.

66. It is clear that $\mu_\alpha, \mu_\beta, \ldots \mu_\epsilon$ will be susceptible of $\frac{5.4.3}{1.2.3}$, or ten differently derived sets of values. The ten equations which may thus arise I shall for the moment represent by

$$\begin{gathered}
{}^1\mu_\alpha y_\alpha+{}^1\mu_\beta y_\beta+\ldots+{}^1\mu_\epsilon y_\epsilon=0,\\
{}^2\mu_\alpha y_\alpha+{}^2\mu_\beta y_\beta+\ldots+{}^2\mu_\epsilon y_\epsilon=0,\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
{}^{10}\mu_\alpha y_\alpha+{}^{10}\mu_\beta y_\beta+\ldots+{}^{10}\mu_\epsilon y_\epsilon=0.
\end{gathered}$$

All these will belong to the same system.

67. Further, if we consider that $y_\alpha+y_\beta+y_\gamma+y_\delta+y_\epsilon$, which must be equated to zero, will give

$$\begin{aligned}
x_\alpha^3+x_\beta^3+\ldots+x_\epsilon^3+(x_\alpha^2+x_\beta^2+\ldots+x_\epsilon^2)p_1&\\
+(x_\alpha+x_\beta+\ldots+x_\epsilon)p_2+5p_3&=0;
\end{aligned}$$

and that consequently, if we eliminate p_3 from the equation

$$x_\tau^3+p_1x_\tau^2+p_2x_\tau+p_3=y_\tau,$$

there will result

$$y_\tau=x_\tau^3-\tfrac{1}{5}\mathfrak{S}3+(x_\tau^2-\tfrac{1}{5}\mathfrak{S}2)p_1+(x_\tau-\tfrac{1}{5}\mathfrak{S}1)p_2;$$

we shall perceive that any two of those ten equations will, if incapable of being made coincident by any transformation and reduction, be sufficient for enabling us to express p_1 and p_2 as rational functions of the roots of the original equation.

68. It is not difficult actually to obtain such a pair of equations, and to show that those values of p_1, p_2, which satisfy them,

will necessarily satisfy every other pair belonging to the same system.

In effect, if introducing an indeterminate multiplier κ we unite

$$0=\kappa(y_\alpha+y_\beta+y_\gamma+y_\delta+y_\epsilon)$$

with

$$\Phi=\mu_\alpha y_\alpha+\mu_\beta y_\beta+\mu_\gamma y_\gamma+\mu_\delta y_\delta+\mu_\epsilon y_\epsilon,$$

or dividing the result by $\mu_\alpha+\kappa$, and designating

$$\frac{\mu_\tau+\kappa}{\mu_\alpha+\kappa}$$

by

$$\nu_\tau,$$

we shall find

$$\frac{\Phi}{\mu_\alpha+\kappa}=y_\alpha+\nu_\beta y_\beta+\nu_\gamma y_\gamma+\nu_\delta y_\delta+\nu_\epsilon y_\epsilon.$$

Now in order that Φ may be different from

$$\mu_\alpha(y_\alpha+y_\beta+y_\gamma+y_\delta+y_\epsilon),$$

κ must admit of being determined so as to satisfy at least one equation of the form

$$\mu_\upsilon+\kappa=0,$$

without causing $\mu_\alpha+\kappa$ to vanish. If, therefore, we reflect that the system of equations on which Φ depends will remain unaltered if, while we substitute another imaginary root ι' for ι, we make certain substitutions among y_β, y_γ, y_δ, y_ϵ, and that consequently Φ may be deduced from

$$\frac{\Phi}{\mu_{\alpha'}+\kappa'}=y_\alpha+\nu_{\beta'}y_{\beta'}+\nu_{\gamma'}y_{\gamma'}+\nu_{\delta'}y_{\delta'}+\nu_{\epsilon'}y_{\epsilon'},$$

$\nu_{\upsilon'}$ being the same function of ι' and κ' as ν_υ is of ι and κ, but $y_{\upsilon'}$ being a different root from y_υ, we shall readily perceive that the coefficient of *any one indifferently* of the four roots $y_\beta, y_\gamma, y_\delta, y_\epsilon$ may be equated to zero, when the coefficient of y_α is equal to 1.

Accordingly, let us suppose that

$$\nu_\epsilon=0;$$

and, on expressing y_α, y_β, . . y_δ in terms of t and u, there will arise

$$\frac{\Phi}{\mu_\alpha-\mu_\epsilon}=(1+\iota\nu_\beta+\iota^2\nu_\gamma+\iota^3\nu_\delta)t+(1+\iota^4\nu_\beta+\iota^3\nu_\gamma+\iota^2\nu_\delta)u.$$

This expression for $\frac{\Phi}{\mu_\alpha - \mu_\epsilon}$ must, in vanishing, assume the form $0t + 0u$; for $\frac{t}{u}$, which is not independent of A'_2, A'_5, cannot generally be equal to $-\frac{1+\iota^4\nu_\beta + ..}{1+\iota\,\nu_\beta + ..}$. We must therefore have separately,

$$1 + \iota\nu_\beta + \iota^2\nu_\gamma + \iota^3\nu_\delta = 0,$$
$$1 + \iota^4\nu_\beta + \iota^3\nu_\gamma + \iota^2\nu_\delta = 0.$$

If now we multiply the first of these equations by ι^ζ, and from the product subtract the second, we shall find

$$\iota^\zeta - 1 + (\iota^{\zeta+1} - \iota^4)\nu_\beta + (\iota^{\zeta+2} - \iota^3)\nu_\gamma + (\iota^{\zeta+3} - \iota^2)\nu_\delta = 0;$$

where ζ may have an unlimited number of different values assigned to it.

Hence, if we cause ν_γ to disappear by making

$$\zeta + 2 = 3,$$

there will result

$$\nu_\beta = \nu_\delta + a_\iota,$$

a_ι denoting

$$-\frac{\iota - 1}{\iota^2 - \iota^4};$$

and if

$$\zeta + 1 = 4,$$

we shall have

$$\nu_\gamma = a_\iota \nu_\delta + 1,$$

$-\frac{\iota - \iota^2}{1 - \iota^3}$, the coefficient of ν_δ being evidently equal to $\frac{\iota}{\iota}\,a_\iota$.

Finally, on returning to the expression for $\frac{\Phi}{\mu_\alpha + \kappa}$, which involves y_α, y_β.., and making the requisite substitutions in it, we shall obtain

$$\frac{\Phi}{\mu_\alpha - \mu_\epsilon} = y_\alpha + y_\gamma + a_\iota\, y_\beta + (y_\beta + y_\delta + a_\iota\, y_\gamma)\nu_\delta;$$

which, if $\Phi = 0$, will give, independently of ν_δ,

$$\left.\begin{aligned} y_\alpha + y_\gamma + a_\iota\, y_\beta = 0, \\ y_\beta + y_\delta + a_\iota\, y_\gamma = 0. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad \text{(b)}$$

And we see that the values of p_1 and p_2 which satisfy this pair of equations must be such as to fulfil the condition $\Phi = 0$, or

$$\mu_\alpha\, y_\alpha + \mu_\beta\, y_\beta + \mu_\gamma\, y_\gamma + \mu_\delta\, y_\delta + \mu_\epsilon\, y_\epsilon = 0.$$

69. We might now, by means of the equations (b.) which involve p_1, p_2, x_{α}, x_{β}, x_{γ}, x_{δ}, and which are both of them of the first degree with respect to p_1 and p_2, express p_1 and p_2 as rational functions of x_{α}, x_{β}, x_{γ}, x_{δ}; and then from discovering the number of different values which the expression for p_n (either p_1 or p_2) would assume if the five indices 1, 2, 3, 4, 5 were made to enter into it four at a time in every order of succession, determine the degree of the final equation $\chi_n(p_n, A_1, A_2, \ldots A_5)=0$; χ_n representing a rational function. Certain properties of the roots of this equation would also become known. But we shall arrive far more rapidly at the same results, from considering the ten equations in question; the remaining eight of which are connected by a remarkable law with those already found.

70. It is obvious that, with a fixed expression for Φ, we should have obtained a different pair of equations if, instead of supposing that $\nu_{\epsilon}=0$, we had equated another of the coefficients ν_{β}, ν_{γ}, ν_{δ}, ν_{ϵ} to zero. It will not, however, be necessary for us to retrace our steps in order to complete the system. From either of the equations (b.) we may discover all the rest. Thus if we take the first of them, and represent by

$$y_{\alpha}' + y_{\gamma}' + a_{\iota'} y_{\beta}' = 0$$

what that equation will become if ι' be substituted for ι and the system (a.) remain unaltered, we shall see that, ι' being different from ι, there will arise a new equation belonging to the system. A difficulty here indeed presents itself. For if we write $\iota^n t'$ for t, and $\iota^{4n} u'$ for u, it will be evident that we may obtain, corresponding to each of the four expressions a_{ι}, a_{ι^2}, a_{ι^3}, a_{ι^4}, five equations of the form in question. We appear, therefore, at first view to be conducted to twenty, and not ten equations in the system. But an examination of the function $a_{\iota'}$ will, as I proceed to show, point out the relation

$$a_{\iota^n} = a_{\iota^{4n}};$$

which includes the two conditions

$$a_{\iota} = a_{\iota^4}, \quad a_{\iota^2} = a_{\iota^3}.$$

71. Reverting to the expression for a_{ι}, we see that

$$(\iota^2 - \iota^4) a_{\iota} = 1 - \iota.$$

Now a_ι, considered generally as a rational function of ι, may evidently be included under the form

$$a_\iota = c_4 + c_3\,\iota + c_2\,\iota^2 + c_1\,\iota^3 + c_0\,\iota^4\,;$$

where $c_4, c_3, \ldots c_0$ do not involve ι.

Hence we find

$$1-\iota = c_1 - c_3 + (c_0 - c_2)\iota + (c_4 - c_1)\iota^2 + (c_3 - c_0)\iota^3 + (c_2 - c_4)\iota^4,$$

which will be satisfied independently of ι, if

$$c_1 - c_3 = 1,\quad c_0 - c_2 = -1,$$
$$c_4 - c_1 = 0,\quad c_3 - c_0 = 0,\quad c_2 - c_4 = 0.$$

We thus perceive that

$$\begin{aligned} a_\iota &= c_1(1+\iota^2+\iota^3) + c_0(\iota+\iota^4) \\ &= 1+\iota^2+\iota^3 + c_0(1+\iota+\iota^2+\iota^3+\iota^4) = -(\iota+\iota^4). \end{aligned}$$

From which there will result, on writing ι^n instead of ι,

$$a_{\iota^n} = a_{\iota^{4n}}.$$

We might have arrived at the expression for a_ι from considering that $y_\alpha + y_\gamma + a_\iota y_\beta$, which, expressed as a function of t and u, would become $\iota^4(\iota^4+\iota+a_\iota)t + \iota(\iota+\iota^4+a_\iota)u$, must in vanishing assume the form $0t+0u$.

It appears, then, that a_{ι^n} has only two different values, a_ι and a_{ι^2}. Hence all functions symmetrical relatively to a_ι and a_{ι^2} will remain unaltered when for ι we substitute any one of the three roots ι^2, ι^3, ι^4.

And in accordance with this we find

$$a_\iota + a_{\iota^2} = -(\iota+\iota^2+\iota^3+\iota^4) = -(-1),$$
$$a_\iota \times a_{\iota^2} = (-1)^2(\iota+\iota^4)(\iota^2+\iota^3) = (-1)^2(-1).$$

a_ι and a_{ι^2} are in fact the roots of the equation

$$a^2 - a - 1 = 0\,;$$

which, solved as a quadratic equation, will give

$$a = \frac{1 \pm \sqrt{5}}{2}.$$

72. Another consequence of the properties of a_ι must here be pointed out.

Representing any one of the ten equations of the system by

$$y_{\mathrm{a}} + y_{\mathrm{c}} + a_{\iota^n}\, y_{\mathrm{b}} = 0,$$

and observing that

$$a_{\iota n} = 1 - a_{\iota 2n},$$

we see that

$$\begin{aligned} y_{\mathrm{a}} + y_{\mathrm{c}} + a_{\iota n}\, y_{\mathrm{b}} &= y_{\mathrm{a}} + y_{\mathrm{b}} + y_{\mathrm{c}} - a_{\iota 2n}\, y_{\mathrm{b}} \\ &= -(y_{\mathrm{d}} + y_{\mathrm{e}} + a_{\iota 2n}\, y_{\mathrm{b}})\,; \end{aligned}$$

a, b, .. e having, for greater simplicity, been introduced instead of the accented indices $\alpha^{\backprime}, \beta^{\backprime}, .. \epsilon^{\backprime}$.

Thus it appears that p_1 and p_2 cannot be determined by means of any two equations expressible by

$$\left.\begin{aligned} y_{\mathrm{a}} + y_{\mathrm{c}} + a_{\iota n}\, y_{\mathrm{b}} = 0, \\ y_{\mathrm{d}} + y_{\mathrm{e}} + a_{\iota 2n}\, y_{\mathrm{b}} = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

which, although seemingly independent of each other, are in reality reducible to a single equation.

73. In discussing equations of the class (c) the following definition will be found useful :—

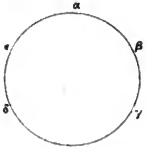
Of the two functions $y_{\mathrm{a}} + y_{\mathrm{c}} + a_{\iota n}\, y_{\mathrm{b}}$, $y_{\mathrm{d}} + y_{\mathrm{e}} + a_{\iota n}\, y_{\mathrm{b}}$, which are of the same form, and which taken together include all the roots of the equation for y, y_{b} remaining fixed, the one will be said to be the *complement* of the other; and either of these functions with the letter c prefixed to it as a characteristic will express symbolically the complement of that function.

74. It may now be seen, either from multiplying $y_\alpha, y_\beta, .. y_\epsilon$ successively by a_ι, or from the law of the indices in the equations (b), that the ten equations,

$$\left.\begin{aligned} y_\epsilon + y_\beta - (\iota + \iota^4) y_\alpha = 0, \\ y_\alpha + y_\gamma - (\iota + \iota^4) y_\beta = 0, \\ y_\beta + y_\delta - (\iota + \iota^4) y_\gamma = 0, \\ y_\gamma + y_\epsilon - (\iota + \iota^4) y_\delta = 0, \\ y_\delta + y_\alpha - (\iota + \iota^4) y_\epsilon = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(d)}$$

$$\left.\begin{aligned} c(y_\epsilon + y_\beta - (\iota^2 + \iota^3) y_\alpha) = 0, \\ c(y_\alpha + y_\gamma - (\iota^2 + \iota^3) y_\beta) = 0, \\ c(y_\beta + y_\delta - (\iota^2 + \iota^3) y_\gamma) = 0, \\ c(y_\gamma + y_\epsilon - (\iota^2 + \iota^3) y_\delta) = 0, \\ c(y_\delta + y_\alpha - (\iota^2 + \iota^3) y_\epsilon) = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(e)}$$

where the indices in each vertical column follow the same order of succession as in the cycle



cannot, while $\alpha, \beta, .. \epsilon$ remain unaltered, conduct to more than one set of values for p_1 and p_2. But before discussing the equations (d) and (e), I proceed to consider properties of functions in which the elements are supposed to change places among themselves.

§ 2.

Of Substitutions.

75. Let X represent a function of n independent quantities $x_a, x_b, x_c, ..$; then $X\begin{pmatrix} x_a \, x_b \, x_c \, .. \\ x_\alpha \, x_\beta \, x_\gamma \end{pmatrix}$ or simply $X\begin{pmatrix} a \, b \, c \, .. \\ \alpha \, \beta \, \gamma \end{pmatrix}$ will express, according to a known notation, that in the function X the quantities $x_a, x_b, x_c, ..$ have been changed into $x_\alpha, x_\beta, x_\gamma, ..$ respectively. I shall term $\begin{pmatrix} a \, b \, c \, .. \\ \alpha \, \beta \, \gamma \end{pmatrix}$ the affix of substitution.

76. The number of different values which X can receive when we change the order of the elements on which it depends, cannot exceed the product $1 . 2 . 3 . . n$; but the affix of substitution will admit of $1 . 2 . 3 . . n \times 1 . 2 . 3 . . n$ differently derived expressions.

77. Thus if we denote by $\Lambda_1, \Lambda_2, .. \Lambda_{1.2.3..n}$ the different forms or states which the indices $(1, 2, 3 .. n)$ are capable of assuming from the several changes of arrangement to which they are supposed to be subjected, the values of X may all of them be expressed by

$$X\begin{pmatrix}\Lambda_1\\\Lambda_1\end{pmatrix}, \quad X\begin{pmatrix}\Lambda_1\\\Lambda_2\end{pmatrix}, \quad X\begin{pmatrix}\Lambda_1\\\Lambda_3\end{pmatrix}, .. X\begin{pmatrix}\Lambda_1\\\Lambda_\nu\end{pmatrix},$$

ν denoting the product $1 . 2 . 3 . . n$; but in this system we may successively substitute $\Lambda_2, \Lambda_3, .. \Lambda_\nu$ for Λ_1: whence will result $(\nu - 1)$ other systems, each of them consisting of ν terms.

78. Suppose X to be such that the number of different values of which it is susceptible shall be less than ν.

Here certain terms in the system

$$X\binom{\Lambda_1}{\Lambda_1},\quad X\binom{\Lambda_1}{\Lambda_2},\ldots X\binom{\Lambda_1}{\Lambda_\nu}$$

must be equal to each other.

Let therefore

$$X\binom{\Lambda_1}{\Lambda_1}=X\binom{\Lambda_1}{\Lambda_2}=\ldots=X\binom{\Lambda_1}{\Lambda_\mu}.$$

On submitting each of these μ expressions to the substitution denoted by $\binom{\Lambda_1}{\Lambda_{\mu+1}}$, and observing that instead of an expression of the form

$$X\binom{\Lambda_1}{\Lambda_\tau}\binom{\Lambda_1}{\Lambda_{\mu+1}},$$

where X has been subjected to two successive substitutions, we may write

$$X\binom{\Lambda_1}{\Lambda_v},$$

we shall have no difficulty in perceiving that the new set of equal quantities which will arise may be represented by

$$X\binom{\Lambda_1}{\Lambda_{\mu+1}}=X\binom{\Lambda_1}{\Lambda_{\mu+2}}=\ldots=X\binom{\Lambda_1}{\Lambda_{2\mu}};$$

which quantities are different from the former, but equal to them in number.

If we operate in the same manner with $\binom{\Lambda_1}{\Lambda_{2\mu+1}}$, $\binom{\Lambda_1}{\Lambda_{3\mu+1}}$, .. $\binom{\Lambda_1}{\Lambda_{(\omega-1)\mu+1}}$ until we have exhausted all the substitutions, we shall find that the ν values of X will be separated into ω groups composed each of them of μ terms.

Hence the number of different values which a function of n quantities may receive from all the possible substitutions of these quantities among themselves, is necessarily a submultiple of the product $1 . 2 . 3 .. n$*, as is well known.

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* Taking, for example,

$$n=4,\quad X=x_1x_2+x_3x_4,\qquad \text{(see 29)}$$

we find $\mu=8$, and $\omega=3$.

79. If X be effected by a series of *contiguous** substitutions,

$$\binom{\Lambda_1}{\Lambda_2},\ \binom{\Lambda_2}{\Lambda_3},\ \binom{\Lambda_3}{\Lambda_4},\ldots\binom{\Lambda_{\mu-1}}{\Lambda_\mu},$$

we shall have universally

$$X\binom{\Lambda_1}{\Lambda_2}\binom{\Lambda_2}{\Lambda_3}\binom{\Lambda_3}{\Lambda_4}\ldots\binom{\Lambda_{\mu-1}}{\Lambda_\mu}=X\binom{\Lambda_1}{\Lambda_\mu}.$$

This is evident.

80. Let us now consider

$$X\binom{\Lambda_1}{\Lambda_v}\binom{\Lambda_1}{\Lambda_v}\binom{\Lambda_1}{\Lambda_v}\ldots;$$

where the same substitution is supposed to be applied any number of times in succession to the function X.

It is obvious that a limited number, p, of such operations must bring us to an expression equal to X, and that all the expressions previously obtained will then reappear in a periodical manner.

If, in effect, we denote by $X\binom{\Lambda_1}{\Lambda_v}^r$ the value of X which will arise when the substitution designated by $\binom{\Lambda_1}{\Lambda_v}$ has been applied r times, we shall have

$$X\binom{\Lambda_1}{\Lambda_v}^0,\ X\binom{\Lambda_1}{\Lambda_v}^1,\ X\binom{\Lambda_1}{\Lambda_v}^2,\ldots X\binom{\Lambda_1}{\Lambda_v}^{p-1},$$

after which we shall come to the term $X\binom{\Lambda_1}{\Lambda_v}^p$, which, by hypothesis, is equal to X or $X\binom{\Lambda_1}{\Lambda_v}^0$; and consequently if we continue to operate with $\binom{\Lambda_1}{\Lambda_v}$, we shall merely reproduce the same series of p terms disposed in the same order as before.

Thus we shall obtain, in the form of an equation,

$$X\binom{\Lambda_1}{\Lambda_v}^{ap+r}=X\binom{\Lambda_1}{\Lambda_v}^r;$$

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* A term made use of in connexion with substitutions by M. Cauchy, to whom we are indebted for the results in 80, 81, and 82, as well as for the theorem given above in 79.

a and r representing any integers, zero included. What is termed the *degree* of $\begin{pmatrix}\Lambda_1\\ \Lambda_v\end{pmatrix}$ is indicated by p.

81. There is an inference from the equivalence of differently derived affixes of substitution, of which as yet I have made no mention. It is, that every substitution is either circular*, such as

$$\begin{pmatrix}\alpha\,\beta\,\gamma\,..\,\zeta\,\eta\\ \eta\,\alpha\,\beta\,..\,\epsilon\,\zeta\end{pmatrix},$$

or admits of being resolved into two or more independent circular substitutions. I must spare a little space for proving this.

Now as the columns of elements of which any substitution is composed may change places among themselves without disturbing its effect, let us select for the second place among the columns, in accordance with the arrangement in the type or pattern

$$\begin{pmatrix}\alpha\,\beta\,\gamma\,..\,\zeta\,\eta\\ \eta\,\alpha\,\beta\,..\,\epsilon\,\zeta\end{pmatrix},$$

that column the lower element of which is equal to the upper element of the first column. Such a selection is always possible, since each of the two horizontal rows of every substitution is supposed to consist of the same elements abstractedly of the order of position

In this way the first two columns of any substitution may be

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* The applicability of the word circular, apart from the form of the function operated on, is not perhaps very easily seen. But let

$$X=\Psi(\alpha,\beta,\gamma,..\zeta,\eta),$$

and there will result

$$X\begin{pmatrix}\alpha\,\beta\,\gamma\,..\,\zeta\,\eta\\ \eta\,\alpha\,\beta\,..\,\epsilon\,\zeta\end{pmatrix}=\Psi(\eta,\alpha,\beta,..\epsilon,\zeta).$$

The effect then of the proposed substitution, the elements $\alpha,\beta,\gamma,..\zeta\;\eta$ being arranged round a circle, as in 74, clearly is to cause each element to advance into the place of the one next preceding it. After a second substitution identical with the first, each of the elements will therefore have moved forward through two places; after a third substitution, through three places; and so on continually. Whence it follows that if p denote the number of the elements, we shall have

$$X\begin{pmatrix}\alpha\,\beta\,\gamma\,..\,\zeta\,\eta\\ \eta\,\alpha\,\beta\,..\,\epsilon\,\zeta\end{pmatrix}^p=\Psi(\alpha,\beta,\gamma,..\zeta,\eta);$$

in which each element has returned to its first position, having completed a revolution round the circle.

represented by

$$\left.\begin{matrix} a & h' \\ h & a \end{matrix}\right\} ; \qquad \ldots \ldots \ldots \quad (1)$$

the element a occurring twice, but no fixed relation existing between h and h'.

Should the proposed substitution be such as to cause the element h of the upper row to come into the place occupied in (1) by h', a circular substitution of the required type

$$\begin{pmatrix} a & h \\ h & a \end{pmatrix} \qquad \ldots \ldots \ldots \quad (1)'$$

would be at once arrived at.

When, however, h', as will more frequently happen, merely represents some one of the elements different from h, we must proceed precisely as before, and select our third column so that we shall have

$$\left.\begin{matrix} a & h' & h'' \\ h & a & h' \end{matrix}\right\} ; \qquad \ldots \ldots \ldots \quad (2)$$

remembering that the lower element of each new column must always be taken equal to the upper element of the preceding one.

Here, if $h''=h$, there will arise a circular substitution with three elements in each horizontal row,

$$\begin{pmatrix} a & h' & h \\ h & a & h' \end{pmatrix}. \qquad \ldots \ldots \ldots \quad (2)'$$

And if $h'' >$ or $< h$, by adding in succession, if required, a fourth, a fifth, a uth column, we shall, in every case, ultimately come to a column in which the upper element is equal to h.

To conclude, when we have reached the uth column, if we find that we have exhausted all the columns in which the upper and lower elements are different, it will be seen that $\begin{pmatrix} \Lambda_1 \\ \Lambda_v \end{pmatrix}$ is, in the particular case under discussion, itself circular. But if beyond the uth column there remain columns of unequal elements, the proposed substitution must manifestly be such as to separate into a finite number of circular substitutions of the prescribed type*.

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* Thus the operation designated by

$$\begin{pmatrix} a & b & c & d & e & f & g & h & i & j & o \\ h & o & d & f & b & j & a & g & e & c & i \end{pmatrix}$$

is equivalent to the three independent circular operations,

$$\begin{pmatrix} a & g & h \\ h & a & g \end{pmatrix}, \quad \begin{pmatrix} b & e & i & o \\ o & b & e & i \end{pmatrix}, \quad \begin{pmatrix} c & j & f & d \\ d & c & j & f \end{pmatrix}.$$

82. An important theorem on the decomposition of substitutions here presents itself.

Observing that if

$$\begin{pmatrix}\Lambda_1\\ \Lambda_{v'}\end{pmatrix}=\begin{pmatrix}\alpha & \beta & \gamma & .. & \zeta & \eta\\ \beta & \alpha & \gamma & .. & \zeta & \eta\end{pmatrix},$$

we shall have

$$\mathrm{X}\begin{pmatrix}\Lambda_1\\ \Lambda_{v'}\end{pmatrix}=\mathrm{X}\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix},$$

where $\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}$ indicates an *interchange* or *transposition* of the elements α and β; and that if

$$\begin{pmatrix}\Lambda_1\\ \Lambda_{v''}\end{pmatrix}=\begin{pmatrix}\alpha & \beta & \gamma & \delta & .. & \zeta & \eta\\ \gamma & \alpha & \beta & \delta & .. & \zeta & \eta\end{pmatrix},$$

we shall have

$$\mathrm{X}\begin{pmatrix}\Lambda\\ \Lambda_{v''}\end{pmatrix}=\mathrm{X}\begin{pmatrix}\alpha & \beta & \gamma\\ \gamma & \alpha & \beta\end{pmatrix}$$

$$=\mathrm{X}\begin{pmatrix}\alpha & \beta & \gamma\\ \beta & \alpha & \gamma\end{pmatrix}\begin{pmatrix}\beta & \alpha & \gamma\\ \gamma & \alpha & \beta\end{pmatrix}=\mathrm{X}\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}\beta & \gamma\\ \gamma & \beta\end{pmatrix}, \quad \text{(see 79)}$$

the operation denoted by $\begin{pmatrix}\Lambda_1\\ \Lambda_{v''}\end{pmatrix}$ being in this case equivalent to the two interchanges $\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}$ and $\begin{pmatrix}\beta & \gamma\\ \gamma & \beta\end{pmatrix}$ taken in succession: we are at once led to infer that every substitution may be represented by a succession of interchanges.

And, in effect, if

$$\begin{pmatrix}\Lambda_1\\ \Lambda_v\end{pmatrix}=\begin{pmatrix}\alpha & \beta & \gamma & .. & \zeta & \eta\\ \eta & \alpha & \beta & .. & \epsilon & \zeta\end{pmatrix},$$

then will arise

$$\mathrm{X}\begin{pmatrix}\Lambda_1\\ \Lambda_v\end{pmatrix}=\mathrm{X}\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}\beta & \gamma\\ \gamma & \beta\end{pmatrix}\cdot\cdot\begin{pmatrix}\zeta & \eta\\ \eta & \zeta\end{pmatrix};$$

as will be evident on reflecting that

$$\mathrm{X}\begin{pmatrix}\Lambda_1\\ \Lambda_v\end{pmatrix}=\mathrm{X}\begin{pmatrix}\alpha & \beta & \gamma & .. & \zeta & \eta\\ \zeta & \alpha & \beta & .. & \epsilon & \eta\end{pmatrix}\begin{pmatrix}\zeta & \alpha & \beta & .. & \epsilon & \eta\\ \eta & \alpha & \beta & .. & \epsilon & \zeta\end{pmatrix}$$

$$=\mathrm{X}\begin{pmatrix}\alpha & \beta & \gamma & .. & \zeta\\ \zeta & \alpha & \beta & .. & \epsilon\end{pmatrix}\begin{pmatrix}\zeta & \eta\\ \eta & \zeta\end{pmatrix};$$

or that every such substitution of the nth degree*, n being any integer, may be represented by a like substitution of the $(n-1)$th degree followed by an interchange. The proposition being therefore true of any circular substitution must hold universally (81).

83. I proceed to consider some properties of the function

$$X\binom{\alpha\,\beta}{\beta\,\alpha}\binom{\gamma\,\delta}{\delta\,\gamma}\ldots;$$

which, in accordance with the meaning usually attached to the symbol $(\ldots)$, I shall express by

$$X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})\ldots;$$

$(\underset{\cdot\cdot}{\mathrm{a\,b}})$ thus denoting the same thing as $\binom{\mathrm{a\ b}}{\mathrm{b\ a}}$.

84. Beginning with the function $X(\underset{\cdot\cdot}{\alpha\beta})$, it will at once be seen that

$$X(\underset{\cdot\cdot}{\alpha\beta})=X(\underset{\cdot\cdot}{\beta\alpha});$$

$(\underset{\cdot\cdot}{\alpha\beta})$ and $(\underset{\cdot\cdot}{\beta\alpha})$ being equally expressive of an interchange of the elements α and β.

85. Passing to the function $X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})$, we see that if α, β, γ, δ be unequal, we must have

$$X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})=X(\underset{\cdot\cdot}{\gamma\delta})(\underset{\cdot\cdot}{\alpha\beta}). \quad . \quad . \quad . \quad (\alpha)$$

For the interchanges being, according to this hypothesis, independent of each other, it must be indifferent in what order they are taken.

But if the function in question were of the form $X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\beta\gamma})$, in which the element β is common to the interchanges, we should, exclusively of particular cases, alter the value of the function by inverting the order of the interchanges.

Thus if

$$X=\Psi(\alpha,\beta,\gamma),$$

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* See note, p. 59. The degree of a non-circular substitution may be found by determining when the elements in each of the constituent circular substitutions will severally complete an exact number of revolutions. But the property in question, although interesting in itself, is not one of which we shall have occasion to make any use throughout this Essay.

we shall have

$$X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\beta\gamma})=\Psi(\gamma,\ \alpha\ \beta),$$
$$X(\underset{\cdot\cdot}{\beta\gamma})(\underset{\cdot\cdot}{\alpha\beta'})=\Psi(\alpha,\ \gamma,\ \beta)(\underset{\cdot\cdot}{\alpha\beta'}),$$

β' being arbitrary.

Now, if $\Psi(\gamma,\ \alpha,\ \beta)=\Psi(\alpha,\ \gamma,\ \beta)(\underset{\cdot\cdot}{\alpha\beta'})$, we must take $\beta'=\gamma$. There will consequently result

$$X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\beta\gamma})=X(\underset{\cdot\cdot}{\beta\gamma})(\underset{\cdot\cdot}{\alpha\gamma})\,; \quad . \quad . \quad . \quad (\beta)$$

in the second member of which equation the element γ, and not β, is common to the interchanges. α, β, γ are supposed to be unequal. In $X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\alpha\beta'})$ an inversion of the interchanges can take place without disturbing the value of that function.

Further, it is clear from the equation (β), that

$$X(\underset{\cdot\cdot}{\beta\gamma})(\underset{\cdot\cdot}{\alpha\gamma})=X(\underset{\cdot\cdot}{\alpha\gamma})(\underset{\cdot\cdot}{\alpha\beta}),$$
$$X(\underset{\cdot\cdot}{\alpha\gamma})(\underset{\cdot\cdot}{\alpha\beta})=X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\beta\gamma})\,;$$

so that we shall have the three equal expressions

$$X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\beta\gamma}),\quad X(\underset{\cdot\cdot}{\beta\gamma})(\underset{\cdot\cdot}{\alpha\gamma}),\quad X(\underset{\cdot\cdot}{\alpha\gamma})(\underset{\cdot\cdot}{\alpha\beta}), \quad . \quad (\beta')$$

which, if we continue to apply the equation (β), will reappear periodically.

86. With respect to $X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})(\underset{\cdot\cdot}{\epsilon\zeta})$, on denoting it by Y we shall find, since $X_u(\underset{\cdot\cdot}{\epsilon\zeta})^2=X_u$,

$$X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})=Y(\underset{\cdot\cdot}{\epsilon\zeta}).$$

If therefore we substitute* $X(\gamma\delta)(\alpha\beta')$ for $X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})$, and affect with $(\underset{\cdot\cdot}{\epsilon\zeta})$ both members of the equation which will thence arise, we shall have

$$X(\underset{\cdot\cdot}{\gamma\delta})(\underset{\cdot\cdot}{\alpha\beta'})(\underset{\cdot\cdot}{\epsilon\zeta})=Y=X(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\delta})(\underset{\cdot\cdot}{\epsilon\zeta})\,;$$

that is, we can operate with the first and second of the interchanges as if the third did not exist. In like manner it might be shown, that in operating with the second and third we may neglect the first.

And analogous results will be obtainable, whatever may be the number of the interchanges with which X is affected.

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* See (α) and (β).

§ 3.

Consideration of the ten equations (d), (e) *resumed. Theorems* (f), (g), (h).

87. Returning now to the equations (d), and designating them, in the order in which they occur, by $f_\alpha=0, f_\beta=0, \ldots f_\epsilon=0$, we find, on inspection,

$$f_\alpha(\underset{\cdot\cdot}{\alpha\beta})=f_\beta(\underset{\cdot\cdot}{\epsilon\gamma}),$$
$$f_\gamma(\underset{\cdot\cdot}{\alpha\beta})=f_\epsilon(\underset{\cdot\cdot}{\epsilon\gamma}).$$

We also find, as might have been foreseen,

$$f_\beta(\underset{\cdot\cdot}{\alpha\beta})=f_\alpha(\underset{\cdot\cdot}{\epsilon\gamma}),$$
$$f_\epsilon(\underset{\cdot\cdot}{\alpha\beta})=f_\gamma(\underset{\cdot\cdot}{\epsilon\gamma}).$$

And it will have been observed that the two functions involved in any of these four equations are either f_α, f_β, or f_ϵ, f_γ; the indices being in the former case equal to the elements of the given affix $(\underset{\cdot\cdot}{\alpha\beta})$, and in the latter to those of $(\underset{\cdot\cdot}{\epsilon\gamma})$.

Further, we obtain

$$f_\delta(\underset{\cdot\cdot}{\alpha\beta})=f_\delta(\underset{\cdot\cdot}{\epsilon\gamma});$$

which involves the single function f_δ, the index of which, δ, does not enter into either $(\underset{\cdot\cdot}{\alpha\beta})$ or $(\underset{\cdot\cdot}{\epsilon\gamma})$.

We are thus conducted to the equation (see 84)

$$f_{\mathrm{a}}(\underset{\cdot\cdot}{\alpha\beta})=f_{\mathrm{b}}(\underset{\cdot\cdot}{\gamma\epsilon}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

where a and b are such as, abstractedly of the order in which they are arranged, to be restricted to the three sets of values

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}, \quad \left.\begin{matrix}\gamma\\ \epsilon\end{matrix}\right\}, \quad \left.\begin{matrix}\delta\\ \delta\end{matrix}\right\}.$$

88. When α, instead of occurring among the elements to be interchanged, remains fixed, we have

$$f_{\mathrm{b}}(\underset{\cdot\cdot}{\beta\epsilon})=f_{\mathrm{e}}(\underset{\cdot\cdot}{\gamma\delta}), \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

b and e here depending on

$$\left.\begin{matrix}\beta\\ \epsilon\end{matrix}\right\}, \quad \left.\begin{matrix}\gamma\\ \delta\end{matrix}\right\}, \quad \left.\begin{matrix}\alpha\\ \alpha\end{matrix}\right\}.$$

This theorem may either be derived from the preceding one, or obtained directly from the equations (d).

89. Finally, we obtain

$$f_b(\underset{..}{\beta}\gamma)(\underset{..}{\delta}\epsilon)=(f_c)(\underset{..}{\beta}\epsilon)_n; \quad . \quad . \quad . \quad . \quad \text{(h)}$$

where $(\underset{..}{\beta}\epsilon)_n$ must coincide either with $(\underset{..}{\beta}\epsilon)$ or with $(\underset{..}{\gamma}\delta)$, the complementary interchange relatively to f_α. With respect to b and c, if we take b successively equal to

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \epsilon,$$

the corresponding values of c will be, if $(\underset{..}{\beta}\epsilon)_n=(\underset{..}{\beta}\epsilon)$,

$$\alpha, \quad \gamma, \quad \epsilon, \quad \beta, \quad \delta;$$

but if $(\underset{..}{\beta}\epsilon)_n=(\underset{..}{\gamma}\delta)$, they will be

$$\alpha, \quad \delta, \quad \beta, \quad \epsilon, \quad \gamma;$$

the successive values of c being in each case arranged at equal intervals in the cycle formed with the indices of the equations (d) or (e) taken in order.

And a similar theorem will exist for $f_b(\underset{..}{\beta}\delta)(\underset{..}{\gamma}\epsilon)$.

§ 4.

Of the function P. *Mode of representing the twelve roots of* $P^{12}+B_1P^{11}+\,..\,+B_{12}=0$. *Remarkable relations thus indicated.*

90. The final equation on which p_1, p_2, p_3 depend, is of the (1 . 3 . 4)th degree. This result, which will have been foreseen independently of the form of that equation, or of the nature of its roots*, may be deduced anew, in conjunction with some very remarkable properties of the roots in question, from the theorems given in the last section.

Designating by P one of the quantities p_1, p_2 (say p_1), we may easily perceive from the equation (f) that the expression for P, considered as a function of $x_\alpha, x_\beta, \,..\, x_\epsilon$, will be such as to

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* From considering that the equations

$$A'_1=0, \quad A'_3=0, \quad A_4-\frac{1}{5}A'^2_2=0$$

are of the first, third, and fourth degrees relatively to p_1, p_2, p_3.

assume generally all its unequal values while one of the roots x_α remains fixed, and x_β, x_γ, x_δ, x_ϵ undergo among themselves the different changes of arrangement to which they can be subjected. In effect, since the same function of the roots may be evolved from

$$\left.\begin{array}{l} f_\beta(\underset{\cdot\cdot}{\gamma\epsilon})=0, \\ f_\epsilon(\underset{\cdot\cdot}{\gamma\epsilon})=0, \end{array}\right\} \text{ as from } \left.\begin{array}{l} f_\alpha(\underset{\cdot\cdot}{\alpha\beta})=0, \\ f_\gamma(\underset{\cdot\cdot}{\alpha\beta})=0, \end{array}\right\}$$

the interchange $(\underset{\cdot\cdot}{\alpha\beta})$ may be set aside*. It appears therefore already that the equation on which P depends is capable of being resolved into five equal factors, any one of which cannot rise above the (1 . 2 . 3 . 4)th degree.

Again, it is evident from the next theorem (g), that of the four pairs of equations to which all those which include f_α or

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* To show this more clearly, let

$$\mathrm{P}=\mathrm{R}(x_1,\, x_2,\, \ldots\, x_5);$$

then, whatever may be the form of the rational function characterized by R, we see that all the 1 . 2 . 3 . 4 . 5 values of which P is susceptible may be designated by

$$\begin{array}{llll}
\mathrm{P}_1, & \mathrm{P}_2, & \mathrm{P}_3, & \ldots\ \mathrm{P}_{24}, \\
\mathrm{P}_1(\underset{\cdot\cdot}{12}), & \mathrm{P}_2(\underset{\cdot\cdot}{12}), & \mathrm{P}_3(\underset{\cdot\cdot}{12}), & \ldots\ \mathrm{P}_{24}(\underset{\cdot\cdot}{12}), \\
\mathrm{P}_1(\underset{\cdot\cdot}{13}), & \mathrm{P}_2(\underset{\cdot\cdot}{13}), & \mathrm{P}_3(\underset{\cdot\cdot}{13}), & \ldots\ \mathrm{P}_{24}(\underset{\cdot\cdot}{13}), \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot \\
\mathrm{P}_1(\underset{\cdot\cdot}{15}), & \mathrm{P}_2(\underset{\cdot\cdot}{15}), & \mathrm{P}_3(\underset{\cdot\cdot}{15}), & \ldots\ \mathrm{P}_{24}(\underset{\cdot\cdot}{15});
\end{array}$$

those in the first line denoting the 1 . 2 . 3 . 4 values which P can assume while x_1 remains fixed.

If now P_τ represent the general term of the 24 functions P_1, P_2, .. P_{24}, it is evident that $\mathrm{P}_\tau(\underset{\cdot\cdot}{1\beta})$ may be taken to represent that of the 4×24 functions $\mathrm{P}_1(\underset{\cdot\cdot}{12})$, $\mathrm{p}_2(\underset{\cdot\cdot}{12})$, .. $\mathrm{P}_{24}(\underset{\cdot\cdot}{15})$.

But from the equation (f) we learn that when we have assigned to β any one of the four values 2, 3, 4, 5, we shall be able so to select γ, ϵ from the three thus left that

$$\mathrm{P}_\tau(\underset{\cdot\cdot}{1\beta})=\mathrm{P}_\tau(\underset{\cdot\cdot}{\gamma\epsilon}).$$

And since $\mathrm{P}_\tau(\underset{\cdot\cdot}{\gamma\epsilon})$ must, in accordance with the hypothesis respecting P_τ, occur among P_1, P_2, .. P_{24} (for in the formation of these functions are exhausted all the interchanges which can be made with the four indices 2, 3, 4, 5), we are manifestly permitted in our search for unequal values of P to confine our attention to the 24 functions in question.

$y_\epsilon + y_\beta - (\iota + \iota^4)\, y_\alpha$ are reducible,

$$\left.\begin{array}{r} f_\alpha = 0, \\ f_b = 0, \end{array}\right\} \quad \left.\begin{array}{r} f_\alpha = 0, \\ f_b(\underset{\cdot\cdot}{\beta\epsilon}) = 0, \end{array}\right\} \quad \left.\begin{array}{r} f_\alpha = 0, \\ f_e(\underset{\cdot\cdot}{\gamma\delta}) = 0, \end{array}\right\} \quad \left.\begin{array}{r} f_\alpha = 0, \\ f_e(\underset{\cdot\cdot}{\beta\epsilon})(\underset{\cdot\cdot}{\gamma\delta}) = 0, \end{array}\right\}$$

the first will furnish the same expression for P as the fourth, and the second as the third.

And since f_α will not, while α remains fixed, admit of more than $\frac{4 \times 3}{1 \times 2}$ different expressions, the number of different values which may be assigned to P cannot exceed $\frac{4}{2} \times \frac{4 \times 3}{1 \times 2}$.

P therefore will depend on an equation of the twelfth degree, or rather on an equation of the form

$$(\mathrm{P}^{12} + B_1\, \mathrm{P}^{11} + B_2\, \mathrm{P}^{10} + \ldots + B_{12})^{10} = 0\,;$$

in which $B_1, B_2, \ldots B_{12}$ are symmetrical relatively to $x_1, x_2, \ldots x_5$, and may consequently, as is well known, be expressed as rational functions of $A_1, A_2, \ldots A_5$, the coefficients of the original equation.

91. To obtain the roots of $\mathrm{P}^{12} + B_1\, \mathrm{P}^{11} + \ldots = 0$ in terms of $x_1, x_2, \ldots x_5$, let us suppose that

$$\alpha = 1\,;$$

we shall then, by making the following substitutions with β and ϵ,

$$\begin{array}{cc} \begin{pmatrix} \beta & \epsilon \\ 2 & 5 \end{pmatrix}, & \begin{pmatrix} \beta & \epsilon \\ 3 & 4 \end{pmatrix}, \\ \begin{pmatrix} \beta & \epsilon \\ 3 & 5 \end{pmatrix}, & \begin{pmatrix} \beta & \epsilon \\ 2 & 4 \end{pmatrix}, \\ \begin{pmatrix} \beta & \epsilon \\ 4 & 5 \end{pmatrix}, & \begin{pmatrix} \beta & \epsilon \\ 2 & 3 \end{pmatrix}, \end{array}$$

find the six expressions of which f_1 is susceptible,

$$\begin{array}{ll} y_2 + y_5 - (\iota + \iota^4) y_1, & y_3 + y_4 - (\iota + \iota^4) y_1, \\ y_3 + y_5 - (\iota + \iota^4) y_1, & y_2 + y_4 - (\iota + \iota^4) y_1, \\ y_4 + y_5 - (\iota + \iota^4) y_1, & y_2 + y_3 - (\iota + \iota^4) y_1\,; \end{array}$$

or if, for brevity, we designate the first three of these expressions taken vertically by i_1, k_1, l_1, we shall have

$$\begin{array}{ll} i_1, & \varsigma\, i_1, \\ k_1, & \varsigma\, k_1, \\ l_1, & \varsigma\, l_1\,; \end{array}$$

each expression in one column being the complement of the corresponding expression in the other.

By applying $\begin{pmatrix}\alpha & \beta & \gamma & \delta & \epsilon \\ 1 & 2 & 3 & 4 & 5\end{pmatrix}$ to each equation in the system (d), we obtain $i_1=0$ along with four equations, which, from analogy, we shall indicate by $i_2=0$, $i_3=0$, $i_4=0$, $i_5=0$, in the order of their occurrence.

Again, by applying $\begin{pmatrix}\alpha & \beta & \gamma & \delta & \epsilon \\ 1 & 3 & 2 & 4 & 5\end{pmatrix}$ to each equation in the same system, we obtain $k_1=0$ followed by four equations, which, abstractedly of the order in which they present themselves, will be indicated by $k_2=0$, $k_3=0$, $k_4=0$, $k_5=0$; the index of k being made to correspond with the index of the term multiplied by a_ι.

Lastly, by applying $\begin{pmatrix}\alpha & \beta & \gamma & \delta & \epsilon \\ 1 & 4 & 3 & 2 & 5\end{pmatrix}$, we obtain $l_1=0$, and $l_2=0$, $l_3=0$, $l_4=0$, $l_5=0$.

Now, from the theorem (h) there will result

$$i_{b'}(\underset{\cdot\cdot}{23})(\underset{\cdot\cdot}{45})=(\mathfrak{c}i_{c'})(\underset{\cdot\cdot}{25})_{n},$$
$$k_{b''}(\underset{\cdot\cdot}{32})(\underset{\cdot\cdot}{45})=(\mathfrak{c}k_{c''})(\underset{\cdot\cdot}{35})_{n},$$
$$l_{b'''}(\underset{\cdot\cdot}{43})(\underset{\cdot\cdot}{25})=(\mathfrak{c}l_{c'''})(\underset{\cdot\cdot}{45})_{n},$$

the second members of which will, if $b'=b''=b'''=1$, reduce themselves to $\mathfrak{c}i_1$, $\mathfrak{c}k_1$, $\mathfrak{c}l_1$.

If then we observe that (72)

$$\mathfrak{c}f_\tau=-f'_\tau,$$

f'_τ denoting what f_τ becomes when a_ι is changed into $a_{\iota 2}$, we shall readily perceive that the six groups of equations on which, as we have seen, all the different expressions for P depend will be reducible to the three following groups:—

$$\left.\begin{array}{r} i_1=0, \\ i_v=0, \end{array}\right\} \quad \left.\begin{array}{r} i_1=0, \\ i_v(\underset{\cdot\cdot}{25})_n=0, \end{array}\right\} \quad \left.\begin{array}{r} i'_1=0, \\ i'_v=0, \end{array}\right\} \quad \left.\begin{array}{r} i'_1=0, \\ i'_v(\underset{\cdot\cdot}{25})_n=0, \end{array}\right\} \quad . \; . \quad \text{(i)}$$

$$\left.\begin{array}{r} k_1=0, \\ k_v=0, \end{array}\right\} \quad \left.\begin{array}{r} k_1=0, \\ k_v(\underset{\cdot\cdot}{35})_n=0, \end{array}\right\} \quad \left.\begin{array}{r} k'_1=0, \\ k'_v=0, \end{array}\right\} \quad \left.\begin{array}{r} k'_1=0, \\ k'_v(\underset{\cdot\cdot}{35})_n=0, \end{array}\right\} \quad . \; . \quad \text{(k)}$$

$$\left.\begin{array}{r} l_1=0, \\ l_v=0, \end{array}\right\} \quad \left.\begin{array}{r} l_1=0, \\ l_v(\underset{\cdot\cdot}{45})_n=0, \end{array}\right\} \quad \left.\begin{array}{r} l'_1=0, \\ l'_v=0, \end{array}\right\} \quad \left.\begin{array}{r} l'_1=0, \\ l'_v(\underset{\cdot\cdot}{45})_n=0, \end{array}\right\} \quad . \; . \quad \text{(l)}$$

$(\underset{\cdot\cdot}{25})_n$, $(\underset{\cdot\cdot}{35})_n$, $(\underset{\cdot\cdot}{45})_n$ may denote either the three interchanges $(\underset{\cdot\cdot}{25})$, $(\underset{\cdot\cdot}{35})$, $(\underset{\cdot\cdot}{45})$, or $(\underset{\cdot\cdot}{34})$, $(\underset{\cdot\cdot}{24})$, $(\underset{\cdot\cdot}{23})$, the complementary interchanges relatively to i_1, k_1, l_1. For greater uniformity the same index v has been retained throughout the groups; the expression for P being in every case unaffected in value by writing 2, 3, 4, 5 successively instead of v.

92. But for our purpose it will not be necessary from each pair of these equations actually to find an expression for P in terms of $x_1, x_2, \ldots x_5$. If we denote by

$$\mathrm{P}_{f(\underset{\cdot\cdot}{\mathrm{ab}})(\underset{\cdot\cdot}{\mathrm{cd}})\,\ldots}$$

that value of P which is derived from the pair of equations

$$f_a(\underset{\cdot\cdot}{\mathrm{ab}})(\underset{\cdot\cdot}{\mathrm{cd}})\ldots=0, \quad f_v(\underset{\cdot\cdot}{\mathrm{ab}})(\underset{\cdot\cdot}{\mathrm{cd}})\ldots=0,$$

the twelve values of which P is susceptible will be represented by

$$\left.\begin{array}{llll} \mathrm{P}_{i}, & \mathrm{P}_{i(\underset{\cdot\cdot}{25})_n}, & \mathrm{P}_{i'}, & \mathrm{P}_{i'(\underset{\cdot\cdot}{25})_n}, \\ \mathrm{P}_{k}, & \mathrm{P}_{k(\underset{\cdot\cdot}{35})_n}, & \mathrm{P}_{k'}, & \mathrm{P}_{k'(\underset{\cdot\cdot}{35})_n}, \\ \mathrm{P}_{l}, & \mathrm{P}_{l(\underset{\cdot\cdot}{45})_n}, & \mathrm{P}_{l'}, & \mathrm{P}_{l'(\underset{\cdot\cdot}{45})_n}: \end{array}\right\} \quad \ldots\ldots \quad (\mathrm{m})$$

so that without proceeding any further, we may perceive that the roots of the equation $\mathrm{P}^{12}+B_1\,\mathrm{P}^{11}\ldots=0$ must be such as to admit of being distributed into three groups, which are related to each other in a very remarkable manner; the second pair of roots in each group being derivable from the first pair by merely introducing a_ι^2 instead of a_ι. Indeed the groups themselves may all of them be derived from

$$\mathrm{P}_{f}, \quad \mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})_n}, \quad \mathrm{P}_{f'}, \quad \mathrm{P}_{f'(\underset{\cdot\cdot}{\beta\epsilon})_n},$$

which will represent four roots of the equation for P.

§ 5.

Examination of V_F *or* $\mathrm{P}_f+\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})_n}+\mathrm{P}_{f'}+\mathrm{P}_{f'(\underset{\cdot\cdot}{\beta\epsilon})_n}$. *Degree of the equation in* V. *Principle involved. The equation in* W.

93. At first view it might be imagined that, if

$$V_F=\mathrm{P}_f+\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})_n}+\mathrm{P}_{f'}+\mathrm{P}_{f'(\underset{\cdot\cdot}{\beta\epsilon})_n},$$

the equation for V_F would not rise above the third degree. But although the eight functions

$$\begin{array}{ll} V_F, & V_{F'}, \\ V_{F(\underset{\cdot\cdot}{\beta\epsilon})}, & V_{F'(\underset{\cdot\cdot}{\beta\epsilon})}, \\ V_{F(\underset{\cdot\cdot}{\gamma\delta})}, & V_{F'(\underset{\cdot\cdot}{\gamma\delta})}, \\ V_{F(\underset{\cdot\cdot}{\beta\epsilon})(\underset{\cdot\cdot}{\gamma\delta})}, & V_{F'(\underset{\cdot\cdot}{\beta\epsilon})(\underset{\cdot\cdot}{\gamma\delta})}, \end{array}$$

$V_{F'}$ denoting what V_F becomes when f is changed into f', will be necessarily equal to each other; and consequently if f be changed successively into i, k, l, and F into I, K, L, there will arise, the index 1 remaining fixed, eight functions equal to V_I, eight to V_K, and eight to V_L; we must not conclude that $V_{F(1\beta)}$, in which the index 1 is supposed not to be fixed, will be capable of coinciding with one of the three functions V_I, V_K, V_L. It is true that all the roots of the equation $\mathrm{P}^{12}+B_1\,\mathrm{P}^{11}+\,..=0$ may be evolved separately from a single expression while the index 1 remains fixed. But the question here relates to the possibility of evolving them four at a time in a certain definite order. And, in effect, if we examine the function $V_{F(\underset{\cdot\cdot}{\alpha\beta})}$, we shall find that

$$V_{F(\underset{\cdot\cdot}{\alpha\beta})}=\mathrm{P}_{h'(\underset{\cdot\cdot}{\delta\epsilon})_n}+\mathrm{P}_{g'}+\mathrm{P}_{h(\underset{\cdot\cdot}{\delta\epsilon})_n}+\mathrm{P}_g\,; \quad . \quad . \quad . \quad \text{(n)}$$

wherein I suppose $f_\alpha(\underset{\cdot\cdot}{\beta\gamma})$, $f_\beta(\underset{\cdot\cdot}{\beta\gamma})$, .. to be denoted by g_α, g_γ, ..; $f_\alpha(\underset{\cdot\cdot}{\beta\delta})$, $f_\beta(\underset{\cdot\cdot}{\beta\delta})$ .. by h_α, h_δ, . . ; and the accent attached to the g and h to indicate, as before, a change of a_ι into a_{ι^2}*. Whence

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* It may easily be shown that

$$\mathrm{P}_{h'(\underset{\cdot\cdot}{\delta\epsilon})_n}=\mathrm{P}_{f(\underset{\cdot\cdot}{\alpha\beta})}, \quad \mathrm{P}_{g'}=\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})_n(\underset{\cdot\cdot}{\alpha\beta})}.$$

Observing that $h_\alpha=y_\epsilon+y_\delta+a_\iota\,y_\alpha$, and that consequently $h_\alpha(\underset{\cdot\cdot}{\beta\delta})(\underset{\cdot\cdot}{\gamma\epsilon})=\iota h_\alpha$, we immediately find from the theorem (h),

$$\begin{aligned} h_\sigma(\underset{\cdot\cdot}{\beta\delta})(\underset{\cdot\cdot}{\gamma\epsilon}) &= (\iota h_\tau)(\underset{\cdot\cdot}{\delta\epsilon})_n \\ &= -h'_\tau(\underset{\cdot\cdot}{\delta\epsilon})_n\,; \end{aligned}$$

where σ and τ may have more than one set of values assigned to them. Hence the same expression for P may be evolved from

$$\left.\begin{array}{l} h'_{\tau'}(\underset{\cdot\cdot}{\delta\epsilon})_n=0, \\ h'_{\tau''}(\underset{\cdot\cdot}{\delta\epsilon})_n=0, \end{array}\right\} \text{ as from } \left.\begin{array}{l} h_{\sigma'}(\underset{\cdot\cdot}{\beta\delta})(\underset{\cdot\cdot}{\gamma\epsilon})=0, \\ h_{\sigma''}(\underset{\cdot\cdot}{\beta\delta})(\underset{\cdot\cdot}{\gamma\epsilon})=0. \end{array}\right\}$$

We must accordingly have

it is clear that the four roots which compose the function $V_{F(\underset{..}{\alpha\beta})}$ do not all of them belong to one group, but must have come in pairs from two of the three groups (i), (k), (l). Thus by the method of substitutions explained in § 2, we shall be conducted to an equation for V_F of the form

$$(V^{15}+C_1V^{14}+C_2V^{13}+\ldots+C_{15})^8=0,$$

in which $C_1, C_2, \ldots C_{15}$ will be rational functions of $A_1, A_2, \ldots A_5$.

$$\begin{aligned}\mathrm{P}_{h'(\underset{..}{\delta\epsilon})_n}&=\mathrm{P}_{h(\underset{..}{\beta\delta})(\underset{..}{\gamma\epsilon})}\\&=\mathrm{P}_{f(\underset{..}{\gamma\epsilon})}\\&=\mathrm{P}_{f(\underset{..}{\alpha\beta})}.\end{aligned}$$ See theorem (f).

Again, observing that $g_\alpha=y_\epsilon+y_\gamma+a_\iota y_\alpha$, and therefore $g_\alpha(\underset{..}{\beta\gamma})(\underset{..}{\delta\epsilon})=\iota g_\alpha$, we find, as in the preceding case,

$$\begin{aligned}g_\tau(\underset{..}{\beta\gamma})(\underset{..}{\delta\epsilon})&=(\iota g_v)(\underset{..}{\gamma\epsilon})_n\\&=-g'_v(\underset{..}{\gamma\epsilon})_n;\end{aligned}$$

and thence

$$\begin{aligned}\mathrm{P}_{g'(\underset{..}{\gamma\epsilon})_n}&=\mathrm{P}_{g(\underset{..}{\beta\gamma})(\underset{..}{\delta\epsilon})}\\&=\mathrm{P}_{f(\underset{..}{\delta\epsilon})}.\end{aligned}$$

Now if we apply the interchange $(\underset{..}{\gamma\epsilon})$ to the equations thus obtained, reflecting that, since the index n may be suppressed, there will result $\mathrm{P}_{g'(\underset{..}{\gamma\epsilon})_n(\underset{..}{\gamma\epsilon})}=\mathrm{P}_{g'(\underset{..}{\gamma\epsilon})^2}=\mathrm{P}_{g'}$, we shall have

$$\begin{aligned}\mathrm{P}_{g'}&=\mathrm{P}_{f(\underset{..}{\delta\epsilon})(\underset{..}{\gamma\epsilon})}\\&=\mathrm{P}_{f(\underset{..}{\gamma\epsilon})(\underset{..}{\gamma\delta})}\\&=\mathrm{P}_{f(\underset{..}{\alpha\beta})(\underset{..}{\gamma\delta})};\end{aligned}$$

the last of which functions will manifestly be equal to $\mathrm{P}_{f(\underset{..}{\beta\epsilon})_n(\underset{..}{\alpha\beta})}$; since $(\underset{..}{\beta\epsilon})$ and $(\underset{..}{\gamma\delta})$ are complementary relatively to f_α.

That $\mathrm{P}_{h'(\underset{..}{\delta\epsilon})_n}$ and $\mathrm{P}_{g'}$ do not belong to the same group, we may at once convince ourselves from considering that

$$\mathrm{P}_f\begin{pmatrix}\Lambda_\alpha\\\Lambda_1\end{pmatrix}=\mathrm{P}_i,\quad \mathrm{P}_g\begin{pmatrix}\Lambda_\alpha\\\Lambda_1\end{pmatrix}=\mathrm{P}_k,\quad \mathrm{P}_h\begin{pmatrix}\Lambda_\alpha\\\Lambda_1\end{pmatrix}=\mathrm{P}_l;$$

where

$$\begin{pmatrix}\Lambda_\alpha\\\Lambda_1\end{pmatrix}=\begin{pmatrix}\alpha&\beta&\gamma&\delta&\epsilon\\1&2&3&4&5\end{pmatrix}.$$

And in fact if we suppose f to be changed into i,

$\mathrm{P}_{h'(\underset{..}{\delta\epsilon})_n}$ will become $\mathrm{P}_{l'(\underset{..}{45}}$

$\mathrm{P}_{g'}$ . . . . $\mathrm{P}_{k'}$.

And the roots of the equation $V^{15}+C_1V^{14}+..=0$ will be expressible by

$$\left.\begin{array}{lll} V_{I}, & V_{I(\underset{\cdot\cdot}{12})}, & .. \; V_{I(\underset{\cdot\cdot}{15})}, \\ V_{K}, & V_{K(\underset{\cdot\cdot}{12})}, & .. \; V_{K(\underset{\cdot\cdot}{15})}, \\ V_{L}, & V_{L(\underset{\cdot\cdot}{12})}, & .. \; V_{L(\underset{\cdot\cdot}{15})}; \end{array}\right\} \quad . \; . \; . \; . \; . \; . \qquad (o)$$

which, except in particular cases, will be distinct one from another.

94. If, however, we designate by

$$W_{f'(\underset{\cdot\cdot}{ab})(\underset{\cdot\cdot}{cd})\,..}$$

the function

$$(P_f+P_{f'})(\underset{\cdot\cdot}{ab})(\underset{\cdot\cdot}{cd})\;..,$$

which evidently does not admit of more than six different expressions, and observe that

$$V_F=W_{f'}+W_{f'(\underset{\cdot\cdot}{\beta\epsilon})_n},$$

we shall see that the resolution of the equation $V^{15}+C_1V^{14}+..=0$ may be reduced to that of a determinate equation of six dimensions,

$$W^6+D_1W^5+D_2W^4+..+D_6=0,$$

the roots of which will be

$$\left.\begin{array}{ll} W_{i'}, & W_{i'(\underset{\cdot\cdot}{25})_n}, \\ W_{k'}, & W_{k'(\underset{\cdot\cdot}{35})_n}, \\ W_{l'}, & W_{l'(\underset{\cdot\cdot}{45})_n}. \end{array}\right\} \quad . \; . \; . \; . \; . \; . \qquad (p)$$

95. Could we solve this equation for W, the roots of any equation of the fifth degree might be easily obtained. For from the expression for $P_f+P_{f'}$ we might deduce that for P_f or P. p_1, p_2, p_3 would thus become known. And from combining

$$x^5+A_1x^4+A_2x^3+..+A_5=0$$

with

$$x^3+p_1x^2+p_2x+p_3=y,$$

we should, as in 22, be led to

$$x=q_4+q_3y+q_2y^2+q_1y^3+q_0y^4; \quad . \; . \; . \qquad (q)$$

where q_4, q_3, .. q_0 are rational functions of p_1, p_2, p_3, or simply of P; and where

$$y=\rho t+\rho^4u,$$

$(\rho t)^5$ and $(\rho^4 u)^5$ being, as is well known, the roots of the equation

$$(t^5)^2 + A^{\backprime}_5(t^5) - \left(\frac{A^{\backprime}_2}{5}\right)^5 = 0,$$

and consequently admitting of being expressed in terms of $A^{\backprime}_2$ and $A^{\backprime}_5$, which also are rational functions of p_1, p_2, p_3.

SECOND PART OF THE DEMONSTRATION.

§ 6.

Forms of the functions designated by t *and* u. *Of* $\Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4$.

96. We have not hitherto taken into consideration the forms of the functions denoted by t and u.

Now from 94 we perceive that

$$(\rho t)^5 = \mu + \alpha' \sqrt{\nu},$$
$$(\rho^4 u)^5 = \mu + \alpha'' \sqrt{\nu};$$

in which α' and α'' are the roots of the equation $\alpha^2 - 1 = 0$, and

$$\mu = \frac{-A^{\backprime}_5}{2}, \quad \nu = \left(\frac{-A^{\backprime}_5}{2}\right)^2 + \left(\frac{A^{\backprime}_2}{5}\right)^5.$$

Hence we may take

$$(\rho t) \times (\rho^4 u) = \rho^5 \sqrt[5]{\mu^2 - \nu} = -\frac{A^{\backprime}_2}{5}.$$

Again,

$$y = \rho t + \rho^4 u = \rho t - \frac{A^{\backprime}_2}{5\rho t} \cdot \frac{(\rho t)^4}{(\rho t)^4}$$
$$= \rho t - \frac{A^{\backprime}_2}{5(\mu + \alpha' \sqrt{\nu})} (\rho t)^4;$$

and, as might be foreseen,

$$y = \rho^4 u + \rho t = \rho^4 u - \frac{A^{\backprime}_2}{5\rho^4 u} \cdot \frac{(\rho^4 u)^4}{(\rho^4 u)^4}$$
$$= \rho^4 u - \frac{A^{\backprime}_2}{5(\mu + \alpha'' \sqrt{\nu})} (\rho^4 u)^4.$$

The equation (q) may therefore, without altering the root x, be resolved into the two following equations,

$$\left.\begin{array}{l} x = r'_4 + r'_3(\rho t) + r'_2(\rho t)^2 + r'_1(\rho t)^3 + r'_0(\rho t)^4, \\ x = r'_4 + r''_3(\rho^4 u) + r''_2(\rho^4 u)^2 + r''_1(\rho^4 u)^3 + r''_0(\rho^4 u)^4; \end{array}\right\} \quad . \quad . \quad (\mathrm{r})$$

$r'_4, r'_3, \ldots r'_0$ being rational with respect to $q_4, q_3, \ldots q_0, \mu, \sqrt{\nu}, A'_2$; and r' becoming r'' when α' is changed into α''.

97. If, now, in the equation

$$x = r_4 + r_3 v + r_2 v^2 + r_1 v^3 + r_0 v^4,$$

by which is represented either of the equations (r), we write $\iota^{\zeta} v$ instead of v; taking ζ successively equal to 0, 1, 2, 3, 4, we may, if

$$r_\eta v^{4-\eta} = \Theta_{4-\eta},$$

arrive at the system of equations

$$\left.\begin{aligned} x_\alpha &= \Theta_0 + 1\Theta_1 + 1\Theta_2 + 1\Theta_3 + 1\Theta_4, \\ x_\beta &= \Theta_0 + \iota\ \Theta_1 + \iota^2\Theta_2 + \iota^3\Theta_3 + \iota^4\Theta_4, \\ x_\gamma &= \Theta_0 + \iota^2\Theta_1 + \iota^4\Theta_2 + \iota\ \Theta_3 + \iota^3\Theta_4, \\ x_\delta &= \Theta_0 + \iota^3\Theta_1 + \iota\ \Theta_2 + \iota^4\Theta_3 + \iota^2\Theta_4, \\ x_\epsilon &= \Theta_0 + \iota^4\Theta_1 + \iota^3\Theta_2 + \iota^2\Theta_3 + \iota\ \Theta_4; \end{aligned}\right\} \quad . \quad . \quad . \quad (\theta)$$

from which there will result

$$\begin{aligned} \iota^{5n} x_\alpha &= \ldots + \iota^{(5+0)n}\Theta_n + \ldots, \\ \iota^{4n} x_\beta &= \ldots + \iota^{(4+1)n}\Theta_n + \ldots, \\ \iota^{3n} x_\gamma &= \ldots + \iota^{(3+2)n}\Theta_n + \ldots, \\ \iota^{2n} x_\delta &= \ldots + \iota^{(2+3)n}\Theta_n + \ldots, \\ \iota^{n}\ x_\epsilon &= \ldots + \iota^{(1+4)n}\Theta_n + \ldots; \end{aligned}$$

the successive coefficients of Θ_{n+h} being ι^{5n}, ι^{5n+h}, ι^{5n+2h}, ι^{5n+3h}, ι^{5n+4h}, or $(\iota^h)^0$, $(\iota^h)^1$, $(\iota^h)^2$, $(\iota^h)^3$, $(\iota^h)^4$.

And if we reflect that the system (θ) will remain unaltered if, substituting $\iota^a\Theta_1$, $\iota^{2a}\Theta_2$, $\iota^{3a}\Theta_3$, $\iota^{4a}\Theta_4$, for $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ in the order of their occurrence, we make suitable substitutions among $x_\alpha, x_\beta, x_\gamma, x_\delta, x_\epsilon$; and that Θ_n, or $r_{4-n} v^n$ must be such as to admit of being equated either to $r'_{4-n}(\rho t)^n$ or to $r''_{4-n}(\rho^4 u)^n$; we shall have little difficulty in perceiving that

$$\left.\begin{aligned} &\left[\Theta_n^5 - \frac{1}{5^5}\left(x_\alpha + \iota^{4n} x_\beta + \iota^{3n} x_\gamma + \iota^{2n} x_\delta + \iota^{n}\ x_\epsilon\right)^5\right] \times \\ &\left[\Theta_n^5 - \frac{1}{5^5}\left(x_\alpha + \iota^{n}\ x_\beta + \iota^{2n} x_\gamma + \iota^{3n} x_\delta + \iota^{4n} x_\epsilon\right)^5\right] = 0 \end{aligned}\right\} \quad . \quad . \quad (8)$$

98. We are thus permitted to assume

$$\left.\begin{aligned}\Theta_n'^5 &= \frac{1}{5^5}\left(x_\alpha + \iota^{4n}x_\beta + \iota^{3n}x_\gamma + \iota^{2n}x_\delta + \iota^{n}x_\epsilon\right)^5,\\ \Theta_n''^5 &= \frac{1}{5^5}\left(x_\alpha + \iota^{n}x_\beta + \iota^{2n}x_\gamma + \iota^{3n}x_\delta + \iota^{4n}x_\epsilon\right)^5;\end{aligned}\right\} \quad . \quad . \quad \text{(t)}$$

$\Theta_n'^5$, $\Theta_n''^5$ being equal to $[r'_{4-n}(\rho t)^n]^5$, $[r''_{4-n}(\rho^4 u)^n]^5$ respectively.

99. It follows therefore that

$$\Theta_n'^5 = \Theta_{4n}''^5. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(u)}$$

Whence we deduce

$$\Theta_1'^5 = \Theta_4''^5, \quad \Theta_2'^5 = \Theta_3''^5,$$
$$\Theta_4'^5 = \Theta_1''^5, \quad \Theta_3'^5 = \Theta_2''^5;$$

observing that $\Theta_{5a+b}^5 = \Theta_b^5$.

§ 7.

Mode of indicating that the expression which may take the place of Θ_n *is a function of* $\mathrm{P}_{f(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\,..}$ *Theorem composed of two branches* (v), (w). *Its hypothetical character. Evolution of particular forms of* $\Theta_{n,f(\underset{\cdot\cdot}{\mathrm{a\,b}})\underset{\cdot\cdot}{\mathrm{c\,d}})\,..}$.

100. Let us now consider Θ_n in relation to the different values of P. To indicate that the expression which may take the place of Θ_n is a function of $\mathrm{P}_{f(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\,..}$, I annex to Θ_n the index of P. Thus

$$\Theta_{n,\,f(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\,..}$$

will denote a function of that value of P, the index of which is $f(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\ldots$

101. It is evident from the equations (s) and (t) that we may have

$$\Theta_{n,\,f(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\,..} =$$
$$\left.\begin{aligned}&\frac{\iota^\zeta}{5}\left(x_\alpha + \iota^{4n}x_\beta + \iota^{3n}x_\gamma + \iota^{2n}x_\delta + \iota^{n}x_\epsilon\right)(\underset{\cdot\cdot}{\mathrm{a\,b}})\,(\underset{\cdot\cdot}{\mathrm{c\,d}})\,..\\ \text{if}\qquad &\Theta_{n,\,f(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\,..} = \iota^\zeta(\Theta'_{n,f})(\underset{\cdot\cdot}{\mathrm{a\,b}})\underset{\cdot\cdot}{\mathrm{c\,d}})\,..\end{aligned}\right\}, \quad \text{(v)}$$

or

$$\left.\begin{array}{l}\Theta_{n,f(\underset{\cdot\cdot}{\mathrm{ab}})(\underset{\cdot\cdot}{\mathrm{cd}})\,..} = \\ \qquad \frac{\iota^{\eta}}{5}\,(x_\alpha + \iota^n x_\beta + \iota^{2n} x_\gamma + \iota^{3n} x_\delta + \iota^{4n} x_\epsilon)\left(\underset{\cdot\cdot}{\mathrm{ab}}\right)\left(\underset{\cdot\cdot}{\mathrm{cd}}\right)\,.. \\ \text{if} \qquad \Theta_{n,f(\underset{\cdot\cdot}{\mathrm{ab}})(\underset{\cdot\cdot}{\mathrm{cd}})\,..} = \iota^{\eta}(\Theta''_{n,f})\left(\underset{\cdot\cdot}{\mathrm{ab}}\right)\left(\underset{\cdot\cdot}{\mathrm{cd}}\right)\,.. \end{array}\right\} . \quad (\mathrm{w})$$

This theorem is remarkable, not only for its hypothetical character, but also for being composed of two branches.

102. If we suppose the operation denoted by $\left(\underset{\cdot\cdot}{\mathrm{ab}}\right)\left(\underset{\cdot\cdot}{\mathrm{cd}}\right)\,..$ to take the particular form $\left(\underset{\cdot\cdot}{\mathrm{ab}}\right)\left(\underset{\cdot\cdot}{\mathrm{ab}}\right)$, there will result from the first branch,

$$\left.\begin{array}{l}\Theta_{n,f} = \frac{\iota^{\zeta_1}}{5}\,(x_\alpha + \iota^{4n} x_\beta + \iota^{3n} x_\gamma + \iota^{2n} x_\delta + \iota^n x_\epsilon) \\ \text{if} \qquad \Theta_{n,f} = \iota^{\zeta_1}\Theta'_{n,f}. \end{array}\right\} . \quad . \quad (\mathrm{v}_1)$$

Further, if, observing that $\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})(\underset{\cdot\cdot}{\gamma\delta})} = \mathrm{P}_f$ (see 88), and consequently $\Theta_{n,f(\underset{\cdot\cdot}{\beta\epsilon})(\underset{\cdot\cdot}{\gamma\delta})} = \Theta_{n,f}$, we suppose $\left(\underset{\cdot\cdot}{\mathrm{ab}}\right)\left(\underset{\cdot\cdot}{\mathrm{cd}}\right)\,..$ to become $\left(\underset{\cdot\cdot}{\beta\epsilon}\right)\left(\underset{\cdot\cdot}{\gamma\delta}\right)$, we shall have

$$\left.\begin{array}{l}\Theta_{n,f} = \frac{\iota^{\zeta_2}}{5}\,(x_\alpha + \iota^n x_\beta + \iota^{2n} x_\gamma + \iota^{3n} x_\delta + \iota^{4n} x_\epsilon) \\ \text{if} \qquad \Theta_{n,f} = \iota^{\zeta_2}(\Theta'_{n,f})\left(\underset{\cdot\cdot}{\beta\epsilon}\right)\left(\underset{\cdot\cdot}{\gamma\delta}\right). \end{array}\right\} . \quad . \quad (\mathrm{v}_2)$$

Lastly, since $\mathrm{P}_{f(\underset{\cdot\cdot}{\alpha\beta})(\underset{\cdot\cdot}{\gamma\epsilon})} = \mathrm{P}_f$, the same branch will give

$$\left.\begin{array}{l}\Theta_{n,f} = \frac{\iota^{\zeta_3 + 4n}}{5}\,(x_\alpha + \iota^n x_\beta + \iota^{2n} x_\gamma + \iota^{3n} x_\delta + \iota^{4n} x_\epsilon) \\ \text{if} \qquad \Theta_{n,f} = \iota^{\zeta_3}(\Theta'_{n,f})\left(\underset{\cdot\cdot}{\alpha\beta}\right)\left(\underset{\cdot\cdot}{\gamma\epsilon}\right). \end{array}\right\} . \quad (\mathrm{v}_3)$$

And analogous results will be obtainable from the other branch (w).

Now we see that (v_1) cannot generally apply to $\Theta''_{n,f}$ (since the condition $\Theta''_{n,f} = \iota^{\zeta}\Theta'_{n,f}$ cannot be satisfied without inducing certain relations among $x_1, x_2, \,..\, x_5$*), but that it will apply to $\Theta'_{n,f}$. On the contrary, (v_2) and (v_3) will be clearly applicable to $\Theta''_{n,f}$, and not to $\Theta'_{n,f}$.

If, in accordance with what has been just proved, we take

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* See the equation (u).

$\zeta_1=\zeta_2=0$, we shall have

$$\left.\begin{aligned}\Theta'_{n,f}&=\frac{1}{5}\,(x_\alpha+\iota^{4n}x_\beta+\iota^{3n}x_\gamma+\iota^{2n}x_\delta+\iota^{n}x_\epsilon),\\ \Theta''_{n,f}&=\frac{1}{5}\,(x_\alpha+\iota^{n}\,x_\beta+\iota^{2n}x_\gamma+\iota^{3n}x_\delta+\iota^{4n}x_\epsilon).\end{aligned}\right\}\quad .\quad .\quad (\mathrm{x})$$

There may also subsist

$$\left.\begin{aligned}\Theta_{n,f(\underset{\cdot\cdot}{\beta\epsilon})}&=\frac{1}{5}\,(x_\alpha+\iota^{n}\,x_\beta+\iota^{3n}x_\gamma+\iota^{2n}x_\delta+\iota^{4n}x_\epsilon),\\ \Theta''_{n,f(\underset{\cdot\cdot}{\beta\epsilon})}&=\frac{1}{5}\,(x_\alpha+\iota^{4n}x_\beta+\iota^{2n}x_\gamma+\iota^{3n}x_\delta+\iota^{n}\,x_\epsilon);\end{aligned}\right\}\quad .\ (\mathrm{y})$$

as is evident.

But having decided upon thus fixing the meanings of $\Theta'_{n,f}$, $\Theta''_{n,f}$, $\Theta'_{n,f(\underset{\cdot\cdot}{\beta\epsilon})}$, $\Theta''_{n,f(\underset{\cdot\cdot}{\beta\epsilon})}$, we must be careful in evolving particular forms of $\Theta_{n,f(\underset{\cdot\cdot}{ab})(\underset{\cdot\cdot}{cd})\,\ldots}$ not to lose sight of the equations (x) and (y). Of this, however, more hereafter.

§ 8.

Of the equation (aa), *the first member of which is a function of* P_f, *the second of* $\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})}$. *Origin of the equations* (ab) *and* (ac). *Their coexistence. Nature of the roots of the equation in* W. *Conclusion respecting the possibility of solving equations of the fifth degree.*

103. Again, if we examine the equation

$$(\Theta_n)\left(\underset{\cdot\cdot}{\beta\epsilon}\right)=\frac{1}{5}\,(x_\alpha+\iota^{n}x_\beta+\iota^{3n}x_\gamma+\iota^{2n}x_\delta+\iota^{4n}x_\epsilon),$$

on designating for the moment $(\Theta_n)\left(\underset{\cdot\cdot}{\beta\epsilon}\right)$ by I_n, we shall perceive that

$$\begin{aligned}&\mathrm{I}_0+1\,\mathrm{I}_1+1\,\mathrm{I}_2+1\mathrm{I}_3+1\mathrm{I}_4=x_\alpha=\Theta_0+1\,\Theta_1+1\,\Theta_2+1\,\Theta_3+1\Theta_4,\\ &\mathrm{I}_0+\iota^4\mathrm{I}_1+\iota^3\mathrm{I}_2+\iota^2\mathrm{I}_3+\iota\,\mathrm{I}_4=x_\beta=\Theta_0+\iota\Theta_1+\iota^2\Theta_2+\iota^3\Theta_3+\iota^4\Theta_4,\\ &\mathrm{I}_0+\iota^2\mathrm{I}_1+\iota^4\mathrm{I}_2+\iota\,\mathrm{I}_3+\iota^3\mathrm{I}_4=x_\gamma=\Theta_0+\iota^2\Theta_1+\iota^4\Theta_2+\iota\,\Theta_3+\iota^3\Theta_4,\\ &\mathrm{I}_0+\iota^3\mathrm{I}_1+\iota\,\mathrm{I}_2+\iota^4\mathrm{I}_3+\iota^2\mathrm{I}_4=x_\delta=\Theta_0+\iota^3\Theta_1+\iota\,\Theta_2+\iota^4\,\Theta_3+\iota^2\Theta_4,\\ &\mathrm{I}_0+\iota\,\mathrm{I}_1+\iota^2\mathrm{I}_2+\iota^3\mathrm{I}_3+\iota^4\mathrm{I}_4=x_\epsilon=\Theta_0+\iota^4\Theta_1+\iota^3\Theta_2+\iota^2\,\Theta_3+\iota\,\Theta_4;\end{aligned}$$

from which will result

$$5\Theta_n = (3+\iota^{2n}+\iota^{3n})\mathrm{I}_n + \{1+2(\iota^{2n}+\iota^{3n})\}\mathrm{I}_{2n} \\ + \{1+2(\iota^{n}+\iota^{4n})\}\mathrm{I}_{3n} + (3+\iota^{n}+\iota^{4n})\mathrm{I}_{4n}; \quad . \quad . \quad . \qquad (\mathrm{z})$$

as the general equation between the systems.

Hence if we denote

$$3+\iota^2+\iota^3, \quad 1+2(\iota^2+\iota^3), \quad 1+2(\iota+\iota^4), \quad 3+\iota+\iota^4,$$

by a, b, c, d,

respectively, we shall have

$$\Theta_1 = \frac{1}{5}(a\mathrm{I}_1+b\mathrm{I}_2+c\mathrm{I}_3+d\mathrm{I}_4),$$

$$\Theta_2 = \frac{1}{5}(d\mathrm{I}_2+c\mathrm{I}_4+b\mathrm{I}_1+a\mathrm{I}_3),$$

$$\Theta_3 = \frac{1}{5}(d\mathrm{I}_3+c\mathrm{I}_1+b\mathrm{I}_4+a\mathrm{I}_2),$$

$$\Theta_4 = \frac{1}{5}(a\mathrm{I}_4+b\mathrm{I}_3+c\mathrm{I}_2+d\mathrm{I}_1).$$

Elevating each of these eight functions to the fifth power, I now express, as in 100, Θ_n as a function of P_f, and I_n as a function of $\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})}$*; and observing that

$$\Theta'_{n,f} = \Theta''_{4n,f}, \qquad \Theta'_{n,f(\underset{\cdot\cdot}{\beta\epsilon})} = \Theta''_{4n,f(\underset{\cdot\cdot}{\beta\epsilon})},$$

$$(\Theta'_n)(\underset{\cdot\cdot}{\beta\epsilon}) = \Theta'_{n.f(\underset{\cdot\cdot}{\beta\epsilon})}, \quad (\Theta''_n)(\underset{\cdot\cdot}{\beta\epsilon}) = \Theta''_{n,f(\underset{\cdot\cdot}{\beta\epsilon})},$$

I find

$$\Theta'^{\,5}_{1,f} + \Theta'^{\,5}_{2,f} + \Theta''^{\,5}_{2,f} + \Theta''^{\,5}_{1,f} =$$

$$\frac{1}{5^5}\left[a\Theta'_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} + b\Theta'_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} + c\Theta''_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} + d\Theta''_{1,f(\underset{\cdot\cdot}{\beta\epsilon})}\right]^5$$

$$+\frac{1}{5^5}\left[d\Theta'_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} + c\Theta''_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} + b\Theta'_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} + a\Theta''_{2,f(\underset{\cdot\cdot}{\beta\epsilon})}\right]^5$$

$$+\frac{1}{5^5}\left[d\Theta''_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} + c\Theta'_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} + b\Theta''_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} + a\Theta'_{2,f(\underset{\cdot\cdot}{\beta\epsilon})}\right]^5$$

$$+\frac{1}{5^5}\left[a\Theta''_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} + b\Theta''_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} + c\Theta'_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} + d\Theta'_{1,f(\underset{\cdot\cdot}{\beta\epsilon})}\right]^5; \quad . \qquad (\mathrm{aa})$$

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* On attaching the index of P to the function Θ_n, we shall find ourselves conducted to the first of the equations (x). It is on this account that, in the leading terms of the equation (aa), Θ' is written for Θ. Had we set out with the expression in terms of the roots x_α, x_β, .. x_ϵ, which is contained in the second of the equations (x), Θ'' must have taken the place of Θ'. This is verified in the equation (aa), which is, as we see, symmetrical with respect to Θ' and Θ''.

of which the first member is a function of P_f; and the second of $P_{f(\underset{\cdot\cdot}{\beta\epsilon})}$ another root of the equation for P.

In this theorem we may change f successively into i, k, l; $f(\underset{\cdot\cdot}{\beta\epsilon})$ successively becoming $i(\underset{\cdot\cdot}{25})$, $k(\underset{\cdot\cdot}{35})$, $l(\underset{\cdot\cdot}{45})$. We may also write $f(\underset{\cdot\cdot}{\beta\epsilon})$ and f' instead of f.

104. If, now, we eliminate $\Theta'_{1,f(\underset{\cdot\cdot}{\beta\epsilon})}$, $\Theta'_{2,f(\underset{\cdot\cdot}{\beta\epsilon})}$, $\Theta''_{1,f(\underset{\cdot\cdot}{\beta\epsilon})}$, $\Theta''_{2,f(\underset{\cdot\cdot}{\beta\epsilon})}$, by means of the equations (see 98)

$$\begin{aligned}
\Theta'_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} &= \left\{r'_3 \sqrt[5]{(\mu+\alpha'\sqrt{\nu})^1}\right\}_{f(\underset{\cdot\cdot}{\beta\epsilon})},\\
\Theta'_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} &= \left\{r'_2 \sqrt[5]{(\mu+\alpha'\sqrt{\nu})^2}\right\}_{f(\underset{\cdot\cdot}{\beta\epsilon})},\\
\Theta''_{1,f(\underset{\cdot\cdot}{\beta\epsilon})} &= \left\{r''_3 \sqrt[5]{(\mu+\alpha''\sqrt{\nu})^1}\right\}_{f(\underset{\cdot\cdot}{\beta\epsilon})},\\
\Theta''_{2,f(\underset{\cdot\cdot}{\beta\epsilon})} &= \left\{r''_2 \sqrt[5]{(\mu+\alpha''\sqrt{\nu})^2}\right\}_{f(\underset{\cdot\cdot}{\beta\epsilon})};
\end{aligned}$$

(the index $f(\underset{\cdot\cdot}{\beta\epsilon})$ being used in the same sense throughout), and bear in mind that

$$(\mu+\alpha'\sqrt{\nu})(\mu+\alpha''\sqrt{\nu})=\mu^2-\nu,$$

we shall have little difficulty in perceiving that the second member of the equation (aa) is reducible to the form

$$\begin{aligned}
\Big\{ N_4 + N_3 \sqrt[5]{(\mu+\alpha'\sqrt{\nu})^1} + N_2 \sqrt[5]{(\mu+\alpha'\sqrt{\nu})^2}&\\
+ N_1 \sqrt[5]{(\mu+\alpha'\sqrt{\nu})^3} + N_0 \sqrt[5]{(\mu+\alpha'\sqrt{\nu})^4} \Big\}_{f(\underset{\cdot\cdot}{\beta\epsilon})}&;
\end{aligned}$$

where $N_4, N_3, \ldots N_0$, with $f(\underset{\cdot\cdot}{\beta\epsilon})$ attached as an index to each of them*; that is to say, $N_{4,f(\underset{\cdot\cdot}{\beta\epsilon})}$, $N_{3,f(\underset{\cdot\cdot}{\beta\epsilon})}$, $\ldots$ $N_{0,f(\underset{\cdot\cdot}{\beta\epsilon})}$ are, in general, determinate and rational relatively to $P_{f(\underset{\cdot\cdot}{\beta\epsilon})}$ and $\sqrt{\nu}_{f(\underset{\cdot\cdot}{\beta\epsilon})}$. Accordingly, if we designate by

$${}_{\zeta}\Xi$$

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* I here suppose, precisely as before, the index $f(\underset{\cdot\cdot}{\beta\epsilon})$, with which the bracket is affected, to denote that in each term within the bracket the function P is to be changed into $P_{f(\underset{\cdot\cdot}{\beta\epsilon})}$. The equation of definition which embraces every case is, in fact,

$$\begin{aligned}
&\phi\{X_{\alpha,u}, X_{\beta,u}, X_{\gamma,u}, \ldots\} =\\
&\phi\{X_{\alpha}, X_{\beta}, X_{\gamma}, \ldots\}, u.
\end{aligned}$$

With respect to the comma outside the bracket in the second member, it may, for still greater simplicity, be in general omitted or understood.

the function

$$N_4 + N_3 \iota^{\zeta} \sqrt[5]{(\mu + \alpha' \sqrt{\nu})^1} + N_2 \iota^{2\zeta} \sqrt[5]{(\mu + \alpha' \sqrt{\nu})^2}$$
$$+ N_1 \iota^{3\zeta} \sqrt[5]{(\mu + \alpha' \sqrt{\nu})^3} + N_0 \iota^{4\zeta} \sqrt[5]{(\mu + \alpha' \sqrt{\nu})^4},$$

on putting ζ successively equal to 0, 1, 2, 3, 4, we shall obtain five functions, ${}_0\Xi$, ${}_1\Xi$, ${}_2\Xi$, ${}_3\Xi$, ${}_4\Xi$; the first of which, ${}_0\Xi$, or rather ${}_0\Xi_{f(\underset{..}{\beta\epsilon})}$, will (irrespectively of the rest[*]) be equal to $\{\Theta_1'^5 + \Theta_2'^5 + \Theta_2''^5 + \Theta_1''^5\}_f$, which, for conciseness, is taken to denote the first member of the equation (aa).

We may thus form an equation of five dimensions,

$$(\Xi - {}_0\Xi)(\Xi - {}_1\Xi) \,..\, (\Xi - {}_4\Xi) = 0\,; \quad . \quad . \quad . \quad \text{(ab)}$$

the coefficients of which, when arranged according to the powers of Ξ, shall be rational functions of $P_{f(\underset{..}{\beta\epsilon})}$. For in consequence of the symmetrical manner in which Θ', Θ'', and therefore α', α'' enter into the calculus, the symbol $\surd$, relatively to the function ν, will, as well as $\sqrt[5]{}$, disappear from the coefficients in question.

105. But $\{\Theta_1'^5 + \Theta_2'^5 + \Theta_2''^5 + \Theta_1''^5\}_f$ is a rational function of P_f. In effect, as

$$(\Theta'_{n,f})^5 = \{(r'_{4-n})^5 (\mu + \alpha' \sqrt{\nu})^n\}_f,$$
$$(\Theta''_{n,f})^5 = \{r''_{4-n})^5 (\mu + \alpha'' \sqrt{\nu})^n\}_f,$$

there will manifestly result

$$\{\Theta_1'^5 + \Theta_2'^5 + \Theta_2''^5 + \Theta_1''^5\}_f =$$
$$\{L_1 + \alpha' \sqrt{M_1} + L_2 + \alpha' \sqrt{M_2}$$
$$+ L_2 + \alpha'' \sqrt{M_2} + L_1 + \alpha'' \sqrt{M_1}\}_f$$
$$= 2\{L_1 + L_2\}_f\,;$$

in which $L_{1,f}$ and $L_{2,f}$ may both of them be rational with respect to P_f.

$\{\Theta_1'^5 + \,..\, + \Theta_1''^5\}_f$ cannot therefore, when the roots $x_1, x_2, \,..\, x_5$ change places among themselves, receive more than twelve different values. If, indeed, we consider that

$$\{\Theta_1'^5 + \Theta_2'^5 + \Theta_2''^5 + \Theta_1''^5\}_f =$$
$$\{\Theta_1'^5 + \Theta_2'^5 + \Theta_2''^5 + \Theta_1''^5\}_{f'},$$ [†];

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[*] That is, without considering whether N_3, N_2, N_1, N_0 severally vanish.

[†] A proof of this theorem may be deduced from the property that P_f

we shall readily perceive that $\{\Theta_1'^5 + \ldots + \Theta_1''^5\}_f$ must admit of becoming a root of a determinate equation of the $\frac{12}{2}$th degree expressible by

$$\left.\begin{array}{l}\{\Xi - \Xi_f\}\{\Xi - \Xi_g\}\{\Xi - \Xi_h\} \times \\ \{\Xi - \Xi_{f(\underset{..}{\beta\,\epsilon})}\}\{\Xi - \Xi_{g(\underset{..}{\gamma\,\epsilon})}\}\{\Xi - \Xi_{h(\underset{..}{\delta\,\epsilon})}\} = 0^*;\end{array}\right\} \quad . \quad \text{(ac)}$$

Ξ_f, Ξ_g, .. $\Xi_{h(\underset{..}{\delta\,\epsilon})}$ representing rational functions of P_f, P_g, .. $P_{h(\underset{..}{\delta\,\epsilon})}$.

106. Comparing the equations (ab) and (ac), we are now conducted to an equation of the form

$$\Xi_f - \mathfrak{r}\{P_{f(\underset{..}{\beta\,\epsilon})}\} = 0,$$

in which $\mathfrak{r}$ is expressive of a rational function; that is, we find

$$a_{11} + a_{10} P_f + a_9 P_f^2 + \ldots + a_0 P_f^{11} =$$
$$b_{11} + b_{10} P_{f(\underset{..}{\beta\,\epsilon})} + b_9 P^2_{f(\underset{..}{\beta\,\epsilon})} + \ldots + b_0 P^{11}_{f(\underset{..}{\beta\,\epsilon})};$$

a_{11}, a_{10}, .. a_0; b_{11}, b_{10}, .. b_0 being symmetrical functions of x_1, x_2, .. x_5.

And comparing this equation with

$$P_f^{12} + B_1 P_f^{11} + B_2 P_f^{10} + \ldots + B_{12} = 0,$$

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becomes $P_{f'}$, that is to say, f becomes f' when ι^2 is substituted for ι. Thus (see the equations (x))

$$\{\Theta'_{1,f}\}^5 = \{\Theta''_{2,f}\}^5 \binom{\iota}{\iota^2} = \{\Theta''_{2,f'}\}^5,$$

$$\{\Theta'_{2,f}\}^5 = \{\Theta'_{1,f}\}^5 \binom{\iota}{\iota^2} = \{\Theta'_{1,f'}\}^5,$$

$$\{\Theta''_{2,f}\}^5 = \{\Theta''_{1,f}\}^5 \binom{\iota}{\iota^2} = \{\Theta''_{1,f'}\}^5,$$

$$\{\Theta''_{1,f}\}^5 = \{\Theta'_{2,f}\}^5 \binom{\iota}{\iota^2} = \{\Theta'_{2,f'}\}^5;$$

for

$$\{\Theta_{n,f}\}^5 \binom{\iota}{\iota^2} = \{(r_{4-n})^5 (\mu + \alpha\sqrt{\nu})^n\}_f \binom{\iota}{\iota^2} = \{\Theta_{n,f\binom{\iota}{\iota^2}}\}^5.$$

Whence springs the equation in the text, the function

$$\{\Theta_1'^5 + \Theta_2'^5 + \Theta_2''^5 + \Theta_1''^5\}_{f'}$$

not being affected in value by any change in the order of the symbols $\Theta_1'^5$, $\Theta_2'^5$, $\Theta_2''^5$, $\Theta_1''^5$.

* The equation (ac) must, in fact, when multiplied by 5^5, be capable of coinciding with the celebrated equation of the sixth degree, by which Vandermonde and Lagrange were stopped in their researches on the solution of algebraical equations of the fifth degree.

there will result

$$\mathrm{P}_f = {}^1\mathrm{r}\{\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})}\};$$

where ${}^1\mathrm{r}$ will represent a rational function.

We must also have, since in the theorem (aa) we are permitted to write $f(\underset{\cdot\cdot}{\beta\epsilon})$ instead of f,

$$\Xi_{f(\underset{\cdot\cdot}{\beta\epsilon})} - \mathrm{r}\{\mathrm{P}_f\} = 0,$$

$$\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})} = {}^1\mathrm{r}\{\mathrm{P}_f\};$$

and therefore

$$\mathrm{P}_f = {}^1\mathrm{r}\Big\{{}^1\mathrm{r}\{\mathrm{P}_f\}\Big\}.$$

107. Similarly, on considering that $\mathrm{P}_{f'}$ may be expressed as a rational function of P_f, and $\mathrm{P}_{f'(\underset{\cdot\cdot}{\beta\epsilon})}$ of $\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})}$ (105), we shall see that

$$\mathrm{P}_f = {}_1\mathbf{R}\{\mathrm{P}_f + \mathrm{P}_{f'}\} = {}_1\mathbf{R}\{W_{f^{\backprime}}\},$$

$$\mathrm{P}_f = {}_2\mathbf{R}\{\mathrm{P}_{f(\underset{\cdot\cdot}{\beta\epsilon})} + \mathrm{P}_{f'(\underset{\cdot\cdot}{\beta\epsilon})}\} = {}_2\mathbf{R}\{W_{f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})}\};$$

and thence

$${}_1\mathbf{R}\{W_{f^{\backprime}}\} = {}_2\mathbf{R}\{W_{f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})}\};$$

${}_1\mathbf{R}$ and ${}_2\mathbf{R}$ representing rational functions.

And combining this equation with

$$W_{f^{\backprime}}^6 + D_1 W_{f^{\backprime}}^5 + \ldots + D_6 = 0,$$

we shall ultimately obtain

$$W_{f^{\backprime}} = {}^1\mathbf{R}\{W_{f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})}\};$$

${}^1\mathbf{R}$ also being expressive of a rational function.

The equation for W will therefore belong to a class of equations of the sixth degree, the resolution of which can, as Abel* has shown, be effected by means of equations of the second and third degrees.

Whence I infer the possibility of solving any proposed equation of the fifth degree by a finite combination of radicals and rational functions (95).

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* In a memoir "Sur une classe particulière d'Equations résolubles algébriquement," Crelle's Journal, vol. iv. p. 131. M. Libri, in his mathematical memoirs, has also discussed the class of equations adverted to in the text.

APPLICATIONS AND REFLECTIONS.

APPLICATION I.

108. What, then, it may be asked, is the element omitted by Ruffini, Abel, and other distinguished mathematicians, who have been led to the conclusion that it is not possible in every case to effect the algebraical resolution of equations of the fifth degree? Let us for a moment consider the nature of the difficulty which had to be overcome. It is clear that an expression for a root of the general equation of the fifth degree must involve radicals characterized by each of the symbols $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[5]{\ }$. If, however, we examine all the solutions which have hitherto been discovered of particular equations of that degree, we shall find that into none of them do cubic radicals enter. A great, if not an impassable barrier, seems at first view to oppose their introduction. For how can cubic radicals arise unless there be a cubic equation? And how can there be a cubic equation unless, in opposition to the well-known theorem of M. Cauchy, the number of different values of a non-symmetrical function of five quantities can be depressed to three? In answering these questions, it is manifest that we cannot fail to detect the element of which we are in search.

109. Now the equation of which

$$W_{f^{\backprime}} + {}^{1}\mathbf{R}\{W_{f^{\backprime}}\}$$

is a root will evidently be of the third degree. For omitting the brackets connected with ${}^{1}\mathbf{R}$, we see that

$${}^{1}\mathbf{R}W_{f^{\backprime}} = {}^{1}\mathbf{R}^{2}W_{f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})} = W_{f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})},$$

the exponent, as is usual, indicating a repetition of an operation; and that consequently (107) the root in question will not be affected by writing $f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})$ instead of $f^{\backprime}$. Just mark what occurs here.

110. We must have

$$\{W_{f^{\backprime}} + {}^{1}\mathbf{R}W_{f^{\backprime}}\}(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\ldots = \{W_{f^{\backprime}} + W_{f^{\backprime}(\underset{\cdot\cdot}{\beta\epsilon})}\}(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\ldots, \tag{ϵ}$$

when $(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{c\,d}})\ldots$ takes the form $(\underset{\cdot\cdot}{\mathrm{a\,b}})(\underset{\cdot\cdot}{\mathrm{a\,b}})$; but not for all values of a, b, c, d, . . : since the method of substitutions explained in § 2 will not generally be applicable to processes based upon the theorem (*v*, *w*), which, in relation to $(\underset{..}{a}\,b)(c\,\underset{..}{d})\,..$, is, we must remember, hypothetical in itself. The equation $(\mathfrak{e})$ belongs, in fact, as will appear in the sequel, to a system of five equations, each of which is separated from the rest by appropriate equations of condition, and gives rise to a cubic factor of the equation of the fifteenth degree in $W_{f'} + W_{f'(\underset{..}{\beta\epsilon})}$ or V_F.

Application II.

111. Again, to test the efficacy of the solution we have obtained of equations of the fifth degree in eliciting mathematical truths, let us take

$$\mathrm{P}^{12} + B_1\,\mathrm{P}^{11} + B_2\,\mathrm{P}^{10} + \,..\, + B_{12} = 0,$$

one of the equations involved in the processes we have gone through, and see what we can discover respecting its nature.

If we consider that $\mathrm{P}_{f'(\underset{..}{\beta\epsilon})}$, $\mathrm{P}_{f'}$ are rational functions of $\mathrm{P}_{f(\underset{..}{\beta\epsilon})}$, P_{f}, using $\mathbf{R}$ as the characteristic, we shall have

$$\mathrm{P}_{f'(\underset{..}{\beta\epsilon})} = \mathbf{R}\mathrm{P}_{f(\underset{..}{\beta\epsilon})} = \mathbf{R}\,{}^{1}\mathfrak{r}\mathrm{P}_{f},$$

$$\mathrm{P}_{f'(\underset{..}{\beta\epsilon})} = {}^{1}\mathfrak{r}\mathrm{P}_{f'}, \quad = {}^{1}\mathfrak{r}\mathbf{R}\mathrm{P}_{f}.$$

Hence the equation of the twelfth degree in P will be such that

$$\mathbf{R}\,{}^{1}\mathfrak{r}\mathrm{P}_{f} = {}^{1}\mathfrak{r}\mathbf{R}\mathrm{P}_{f};$$

and if we further consider that there must subsist an equation analogous to (aa) when each of the remaining roots is combined with P_{f}, we shall find ourselves conducted to another class of equations solved by Abel in the memoir already mentioned.

Application III.

112. Lastly, $\Xi_{f(\underset{..}{\beta\epsilon})}$ being (105) a rational function of $\mathrm{P}_{f(\underset{..}{\beta\epsilon})}$, we obtain (106)

$$\Xi_{f} = \mathfrak{r}^{\backprime}\,\Xi_{f(\underset{..}{\beta\epsilon})},$$

$$\Xi_{f(\underset{..}{\beta\epsilon})} = \mathfrak{r}^{\backprime}\,\Xi_{f};$$

$\mathfrak{r}^{\backprime}$ indicating a rational function.

The equation of the sixth degree in Ξ belongs therefore to the same class as that in W, and must consequently admit of a similar solution*.

Nothing more need be said to show the importance of the algebraical resolution of equations.

113. With respect to equations beyond those of the fifth degree, they can be solved in various ways, as in 107, 111, and 112. In passing the limit of $m=5$ every difficulty disappears. But of this hereafter.

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* See the London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science for June 1845.